Infinite symmetries in the quantum Hall effect

Boris Škorić

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The story so far:
In the beginning the Universe was created. This has made a lot of people very angry and been widely regarded as a bad move.
Many races believe that it was created by some sort of god, though the Jatravartid people of Viltvodle VI believe that the entire Universe was in fact sneezed out of the nose of a being called the Great Green Arkleseizure.
The Jatravartids, who live in perpetual fear of the time they call The Coming of The Great White Handkerchief, are small blue creatures with more than fifty arms each, who are therefore unique in being the only race in history to have invented the aerosol deodorant before the wheel.
However, the Great Green Arkleseizure theory is not widely accepted outside Viltvodle VI and so, the Universe being the puzzling place it is, other explanations are constantly being sought.

Douglas Adams, *The Restaurant at the End of the Universe*

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Master's thesis
B. Škorić
Institute for Theoretical Physics
University of Amsterdam
Valckenierstraat 65
1018 XE Amsterdam, the Netherlands
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Supervisor: prof.dr.ir. F.A. Bais

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Chapter 1

General introduction

> There is a theory that states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable.
>
> Douglas Adams, *The Restaurant at the End of the Universe*

The discovery of the quantum Hall effect (QHE) by von Klitzing, Dorda and Pepper in 1980 [1] had an important impact on physics. The quantisation of conductance in certain two-dimensional electron systems in a strong magnetic field came as a complete surprise. Existing theories could not explain why there are large intervals of the ratio n_e/B where the longitudinal conductance completely disappears, while the Hall conductance is quantised with immense precision in integer multiples of e^2/h. The theory of electronic transport properties had to be drastically improved.

Due to the work of Girvin, Haldane, Halperin, Laughlin, Prange, Pruisken, Thouless and others in the early 1980s, the integer quantum Hall effect is now fairly well understood. It was realised that the combination of a random potential, created by the impurities in a sample, and a strong magnetic field gives rise to a special coexistence of localised and extended electron states. It was shown that when the Fermi level lies in an energy interval devoid of extended states, called a mobility gap, a change in the electron density or the magnetic field can only result in a different occupation of localised states, which does not affect the conductance. In this way the occurrence of conductance quantisation was explained. The explanation for the actual values of the quantised Hall conductance was given by an ingeneous gedanken experiment, proving that a system *with* impurities has either zero conductance or the same conductance as a system with the same amount of electrons but *without* impurities. A more general approach, yielding the fact of quantisation as well as the integer quantised values, was developed later with the use of scaling theory. This method also correctly describes the regions where the conductance is not quantised. (A good review of the early work is given in [3].)

The discovery in 1982 of the *fractional* quantum Hall effect [2], where the Hall conductance is quantised in fractional multiples of e^2/h, again showed that the theories were not complete. In the years that followed, ever more new fractions were measured. In 1983 the famous Laughlin wave functions were introduced [8], describing *incompressible quantum fluids* and explaining the small but experimentally prominent class of fractions $\frac{1}{odd}$. It turned out that the quasiparticle excitations of such fluids are charge/flux-composites with fractional charge and statistics, dubbed *anyons*. The special properties of charge/flux-composites were used by Haldane and Halperin [9, 10] and by Jain [14] in the construction of two so-called hierarchies, sets of Hall fractions for which incompressible ground states could be found. Both hierarchies are able to reproduce all observed fractions, but also yield many fractions that have never been measured.

The striking universality in the manifestation of the quantum Hall effect attracted the attention of many theoretical physicists, not only in solid state physics but also in high energy physics and string theory. The latter were at that time involved with mathematical tools as infinite symmetries and topological invariances. Some of the formalism, like two-dimensional conformal field theory, developed at lightning speed after the famous paper by Belavin, Polyakov and Zamolodchikov [17] in 1984, had already been succesfully applied in statistical mechanics and solid state physics.

This extensive mathematical apparatus of topological field theory and infinite dimensional algebras was applied to the QHE. In several independent ways it was found [20, 26] that a description of incompressible quantum fluids reduces to a theory of chiral edge currents. The symmetry algebras of this edge theory are the infinite-dimensional Kač-Moody and Virasoro algebra. Quantum Hall ground states and the quasiparticle excitations were described in terms of representations of these symmetry algebras. In 1992 Capelli, Zemba and Trugenberger [25] found an even larger symmetry algebra: the so-called $W_{1+\infty}$ algebra, generating area preserving diffeomorphisms. The presence of this algebra in the QHE is no mystery, since the only nontrivial transformations of an incompressible fluid are those that affect the shape but not the area. The representation theory of $W_{1+\infty}$ was worked out by Kač et al. in 1993/94 [23, 24] and was recently applied to the quantum Hall effect [27, 28].

The aim of this master's thesis is to present a review of the applications of infinite-dimensional algebras in the quantum Hall effect and to bridge the gaps between the various approaches that have been taken in recent papers. We will mainly discuss the *universal* properties of the QHE, concentrating on the algebraic properties underlying it and on the classification of Hall fractions with the use of representation theory.

The outline of this thesis is as follows: In chapter 2 we briefly present the experimental data on the quantum Hall effect, highlighting the interesting aspects of these data and showing the sharp contrast with the *classical* Hall effect.

Then we discuss the quantum mechanics of noninteracting and interacting two dimensional electrons in chapters 3 and 4, introducing the key concepts needed to understand the integer and fractional effect. The appearance of anyonic quasiparticle excitations is explained and a description in terms of charge/flux composites is given. Both the Haldane-Halperin and the Jain hierarchy are presented.

The macroscopic equations for the current in the QHE can be described by Chern-Simons theory in a natural way. In chapter 5 we give the Chern-Simons action and its topological properties. It is shown that this action can be written as a conformally invariant action of chiral bosons moving on the (1+1)-dimensional edge of the sample. We recall some of the relevant concepts from conformal field theory and abelian bosonisation in (1+1) dimensions. The spectrum of quasiparticles and Hall fractions is derived from the representation theory of the Kač-Moody algebra $\widehat{U(1)}^{\otimes m}$.

In chapter 6 we demonstrate the presence of the $W_{1+\infty}$ algebra in the QHE. Explicit representations of this algebra are constructed by the attachment of flux tubes to the electrons; it is found that quantum Hall ground states are lowest weight states in these representations. Finally we discuss the representation theory of $W_{1+\infty}$ and its implications for the classification of Hall fractions.

We conclude by summarising and comparing the results obtained from the edge boson theory and the $W_{1+\infty}$ representations.

Chapter 2

The quantum Hall effect

> There is another theory that says this has already happened.
>
> Douglas Adams, *The Restaurant at the End of the Universe*

In this introductory chapter we present a brief review of the experimental data on the quantum Hall effect that have been gathered since its discovery. The uniqueness of the QHE and its fundamental importance for physics are discussed. Then the QHE is compared to the well-known classical Hall effect. As expected, it turns out that a naive semiclassical model cannot account for the observed effects.

The contents of this chapter can be found in the standard reference [3].

2.1 The experiments

A two-dimensional electron system can be experimentally realised in so-called inversion layers. An inversion layer is an interface between a semiconductor and an insulator, with an external electric field perpendicular to the interface that attracts electrons from the semiconductor to the interface. The electrons find themselves in a potential well, created by the interface and the electric field. If the temparature is low enough, the motion perpendicular to the interface is frozen out, leaving only two motional degrees of freedom. The system in which the QHE was originally discovered had Si as the semiconductor and SiO_2 as the insulator. A system with very good properties is GaAs–$Al_xGa_{1-x}As$ (the fraction of Ga atoms replaced by Al is denoted by x), with GaAs playing the role of the semiconductor. The typical temperature for measuring the QHE is about 100 mK.

A small current I (less than 100 μA) is sent through a rectangular sample with connections at its sides as shown in figure (2.1.1). The longitudinal and transverse voltage differences V_L and V_H are measured, as well as the current I. From this one calculates the longitudinal resistance R_L and the Hall resistance R_H,

$$R_L = \frac{V_L}{I} \quad ; \quad R_H = \frac{V_H}{I}. \tag{2.1.1}$$

These resistances represent the linear response of the system to the current, since I is small. In 1980 von Klitzing, Dorda and Pepper [1] discovered that at low temperatures the resistances depend on n_e/B as shown in figure (2.1.2).

In some experimental setups it is possible to vary n_e and B independently. It was found that the resistances only depend on the fraction n_e/B.

Figure 2.1.1: Experimental setup

Figure 2.1.2: The IQHE: $\frac{1}{R_H}$ and R_L as a function of $\nu = \frac{n_e h}{eB}$

There are large intervals of n_e/B where the Hall resistance is constant and V_L drops to zero. The flat pieces of the R_H-curve are often referred to as Hall steps or Hall plateaus. The width (or stability) of a step is defined as the width of the n_e/B-interval where the Hall resistance is constant. At very low temperatures the steps are wide and extremely flat. Von Klitzing, Dorda and Pepper found steps with

$$\frac{1}{R_H} = p\frac{e^2}{h} \qquad p = integer \tag{2.1.2}$$

to an accuracy of less than 5 ppm. In up to date experiments this accuracy is better than 0.3 ppm, and the steps are flat within an error of about 0.01 ppm. The longitudinal resistance can become so small that it is hard to measure. In some experiments an upper bound of $10^{-10}\Omega$ has been found.

In fact, already before the experimental discovery of the effect there were some theoretical indications (Ando, Matsumoto and Uemura 1975, Kawaji et al. 1975) that something like the QHE could occur. However, no one was sure of the validity of these calculations. The discovery and the accuracy of the QHE came as a surprise.

Another big surprise came in 1982, when Tsui, Störmer and Gossard [2] discovered that in extremely clean samples at very low temperatures not only integer numbers occur, but also fractional values

$$\frac{1}{R_H} = \frac{p}{q}\frac{e^2}{h} \qquad p = int,\ q = odd\ int. \tag{2.1.3}$$

For obvious reasons this is called the fractional quantum Hall effect (FQHE). The first fractions that were observed are $p/q = 1/3$ and $2/3$. The plateaus at these fractional values are wide and easily discernible. The accuracy of the quantisation of R_H is less than in the IQHE, but still better than 30 ppm for p/q=1/3 and 2/3. Today many different fractions have been observed. Among these are the following:
$\frac{p}{q} = \frac{1}{3}(1,2,4,5,7,8),\ \frac{1}{5}(1,2,3,4,7,8),\ \frac{1}{7}(1,2,3,4,5,10,11),\ \frac{1}{9}(2,4,5,13),$
$\frac{1}{11}(2,3,4,5,6,7,8),\ \frac{1}{13}(3,4,6,7,8,9),\ \frac{1}{15}(4,7,8),\ \frac{1}{17}(8,9,10)$ and $\frac{9}{19}$.

At many of these fractions, the plateaus are very narrow. For some fractions no plateau is visible at all; only a dip of R_L is measured. As a general rule, the plateau width decreases with increasing denominator.

A most striking feature of the quantisation of R_H is that it is *material independent*, provided that the electrons can be made effectively two-dimensional. All other (nonzero) resistances found in nature are material dependent. In the QHE the Hall resistance depends only on the fundamental constants h and e. It is this aspect of *universality* that makes this phenomenon an important challenge for theoretical physics.

Experimental studies of the temperature- and B-dependence of the transition regions between plateaus in the FQHE have revealed that the quasiparticle/quasihole excitations do not behave as fermions or bosons. These excitations are a new kind of particle with new properties. They are called *anyons*, as they can have *any* statistical behaviour between that of bosons and fermions. Anyons have never been observed in any other physical system. In that respect the fractional QHE is absolutely unique. It is the only testing ground for our theoretical understanding of generalised statistics. In chapter 4 we will present a basic description of anyons and discuss their manifest occurrence in the FQHE.

Another remarkable feature of the QHE is that the fundamental quantity is the resistance, *not the resistivity*, in sharp contrast with the standard situation. The QHE occurs in samples that contain impurities and defects. The resistivity cannot be constant in a whole sample, but still the overall result is that the macroscopic resistance is given by (2.1.3). If it were the other way round, with the resistivity quantised, precision measurements would be impossible, since only the resistance can be measured directly. The resistivity can only be found indirectly, using assumptions about the geometry and sample homogeneity.

Resistance is a quantity describing transport from one edge to another; therefore it is to be expected that the edges play an important role in the QHE.

Also of great importance is the experimental observation that the shape of the sample is not important for the QHE. It does not matter whether the sample is a nice rectangle or something completely different. This is very convenient, for the contacts needed to measure the voltages affect the shape. The geometry-independence of the plateau values suggests that on the plateaus the physics should be invariant under the action of a big symmetry algebra that changes the form of the sample. We will see in chapter 6 that this is the so-called $W_{1+\infty}$ algebra, an infinite-dimensional algebra that generates area preserving diffeomorphisms, and that the set of measured values for R_H can be related to the representation theory of this algebra.

The behaviour of the transition regions between the Hall plateaus is material dependent. The plateaus are wide for materials with low electron mobility and narrow for high mobility. (The electron mobility is a quantity expressing the ability of the average electron to conduct.) This indicates that there is a connection between localisation effects and the appearance of plateaus. This connection will be discussed in chapter 3.

In figure (2.1.2) we see that the longitudinal resistance is proportional to the slope of $1/R_H$. This behaviour will also be discussed in chapter 3.

The disappearance of the longitudinal resistance makes the current dissipationless. In this respect the QHE is similar to superconductivity. High T_c superconductors contain copper-oxide-planes, and there have been suggestions that electrons are trapped in these plains, becoming effectively two-dimensional, just as in the QHE. However, there are big differences. Superconductivity is destroyed by strong magnetic fields, while the QHE is *caused* by them.

The particular combination of the fundamental constants h and e occurring in (2.1.2) gives us an opportunity to measure the fine structure constant α

$$\alpha = \frac{\mu_0 c}{2}\frac{e^2}{h}, \tag{2.1.4}$$

in a way that is independent of experiments in particle accelerators. The accuracy of this measurement is equal to the best achieved accuracy in measuring R_H, since the magnetic permeability of the vacuum μ_0 and the speed of light c are exactly defined numbers; $c := 2.99792458 \cdot 10^8$ m/s , $\mu_0 := 4\pi \cdot 10^{-7}$ H/m. It is remarkable indeed that a condensed matter system allows a very precise determination of a fundamental constant, such as the strength of the electromagnetic interaction α, whose precision measurements up to now belonged exclusively in the realm of high energy physics. Within the uncertainties, the results from QHE measurements agree with the value of α obtained from the magnetic moment of the electron and QED calculations.

Also, the extreme flatness of the Hall steps has been used to define a new resistance standard, based on the QHE. The old standard was based on reference resistors whose resistances tend to drift with a speed of some 0.05 ppm/year. The Hall steps do not drift.

Lots of interesting measurements have not been mentioned above, involving for instance alternating currents, temperature dependence of the resistances, critical temperatures/currents/magnetic fields, magnetic fields *in* the plane of the interface etc. They teach us a lot about the material-dependent microscopic details, but it is beyond the scope of this thesis to discuss these measurements. This thesis will instead concentrate on the *universal* aspects of the QHE.

2.2 The classical Hall effect

Let us first review what we know about the classical Hall effect. When a current-carrying planar medium is placed in a magnetic field perpendicular to the plane, the moving charges experience a Lorentz force perpendicular to the current and the magnetic field. They will move to one side of the medium until their displacement has caused a potential difference that exactly compensates the Lorentz force. This occurrence of a potential difference perpendicular to the current is called the Hall effect. The potential difference is called the Hall potential. From the sign of the Hall potential one can tell whether the main mobile charges are electrons or holes. The magnitude of the Hall potential gives information about the density of mobile charges.

Let us assume that the current-carrying particles are electrons with charge $-e$ and particle density n_e freely moving in the (x, y)-plane, and that the magnetic field $\vec{B}$ is a constant vector pointing in the negative z-direction. We will be interested in the resisivity and conductivity.

An electron with velocity $\vec{v}$ is subject to a force

$$\vec{F} = -e\vec{E} - e\vec{v} \times \vec{B}. \tag{2.2.1}$$

The combined electric and Lorentz force cause a current density $\vec{\jmath}$, given by

$$\vec{\jmath} =: -\sigma_0 \frac{\vec{F}}{e} = \sigma_0 (\vec{E} - \frac{\vec{\jmath} \times \vec{B}}{n_e e}) \tag{2.2.2}$$

where σ_0 is the conductivity of the medium in zero magnetic field. This equation can also be written as

$$\vec{E} = \frac{\vec{\jmath}}{\sigma_0} + \frac{\vec{\jmath} \times \vec{B}}{n_e e} =: \rho\vec{\jmath}. \tag{2.2.3}$$

where ρ is the resistivity tensor. We have

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \sigma_0^{-1} & -B/n_e e \\ B/n_e e & \sigma_0^{-1} \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix} \tag{2.2.4}$$

Inversion of ρ gives the conductivity tensor σ,

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \tag{2.2.5}$$

with

$$\sigma_{xx} = \sigma_{yy} = \frac{\sigma_0}{1 + \left(\frac{\sigma_0 B}{n_e e}\right)^2} \quad , \quad \sigma_{xy} = -\sigma_{yx} = \frac{n_e e}{B}[1 - \frac{\sigma_{xx}}{\sigma_0}] \tag{2.2.6}$$

In the semiclassical theory of conductivity, σ_0 can be expressed in terms of the mean free time between scatterings τ_0, and the effective electron mass m. If an electron feels a force

$\vec{F}$, it will on the average pick up an incremental velocity $\triangle\vec{v} = \vec{F}\tau_0/m$ between each scattering. The total current density is $\vec{\jmath} = -n_e e\triangle\vec{v}$, giving

$$\sigma_0 = \frac{n_e e^2 \tau_0}{m}. \tag{2.2.7}$$

Substitution of (2.2.7) into (2.2.6) gives

$$\sigma_{xx} = \frac{\sigma_0}{1+(\omega_c\tau_0)^2}\,, \quad \sigma_{xy} = \frac{n_e e}{B} - \frac{\sigma_{xx}}{\omega_c\tau_0}. \tag{2.2.8}$$

where ω_c is the Larmor or cyclotron frequency; $\omega_c = \frac{eB}{m}$. It is the angular velocity of an electron freely moving in circles in a magnetic field B. Its inverse gives us the typical time scale.

Let us now compare equation (2.2.8) with the conductivities observed in the QHE. If we assume(!) that the sample is homogeneous, we can write down the following resistivity and conductivity for the (I/F)QHE,

$$\rho = \begin{pmatrix} 0 & -\frac{1}{f}\frac{h}{e^2} \\ \frac{1}{f}\frac{h}{e^2} & 0 \end{pmatrix} \quad ; \quad \sigma = \sigma_H \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_H := f\frac{e^2}{h}. \tag{2.2.9}$$

with $f = \frac{p}{q}$, p =integer, q =odd integer. The quantity σ_H is called the Hall conductivity.

Note that σ and ρ are off-diagonal at the same time. We have $\sigma_{xx} = 0 = \rho_{xx}$, which may seem strange.

We see that for $\tau_0 \to \infty$ the conductivity (2.2.8) takes the off-diagonal form of the quantum Hall conductivity (2.2.9), but does not give the plateaus. Some sort of energy gap could cause τ_0 to go to infinity, but in the semiclassical model it is not clear what would cause such an energy gap.

As expected, the semiclassical model of conductivity cannot explain the QHE. We will need a fully quantised model.

Chapter 3

The integer effect

No-one knows the reason for all this, but it is probably quantum.

Terry Pratchett, *Pyramids*

In this chapter we will discuss the behaviour of nonrelativistic noninteracting electrons in a very simple quantum mechanical model and see that many features of the *integer* QHE can be understood in this way. We will assume that the electron spins are completely polarized, so that their spin degree of freedom can be discarded. This is not an unrealistic assumption, as the magnetic field is very strong. Let us for the moment also make the drastic assumption that there is no Coulomb repulsion between the electrons, which greatly simplifies all calculations. The justification for this simplification is that the interelectron repulsion can be neglected when there is a strong short-range disorder potential.

Our first aim is to derive from general principles the conductance of an electron gas in the absence of impurities. Then we recall the solutions of the Schrödinger equation for one electron in a uniform magnetic field, yielding the well known Landau levels. Finally we introduce impurities and discuss their effect on the electron states and the conductance.

The contents of this chapter can be found for a large part in the standard reference [3]. The discussion presented here is by no means complete, but will suffice to introduce the key concepts.

3.1 The conductivity in a pure sample

Let us say that the electrons move in the infinite (x, y)-plane and that the magnetic field $\vec{B}$ is constant and points in the negative z-direction,

$$\vec{B}(x, y) = -B\vec{e}_z. \tag{3.1.1}$$

Let us also take a constant electric field $\vec{E}$ in the (x, y)-plane. In a frame of reference moving with velocity $\vec{v}$,

$$\vec{v} = \frac{1}{B^2}(\vec{E} \times \vec{B}) \tag{3.1.2}$$

with respect to the laboratory frame, the electric field vanishes. In the absence of an electric field there is no electric current, so the current in the moving frame is zero. Using $\vec{\jmath} = -n_e e\vec{v}$ and (3.1.2) we find that the current in the laboratory frame is given by

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \frac{n_e e}{B} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \tag{3.1.3}$$

This argument depends only on the translational invariance of the system. It does not even depend on the electron-electron interaction. As long as there is translational invariance, the current is perpendicular to the electric field and proportional to the electron density, no matter what that density is. The off-diagonal form of the conductance is exactly what we want in a description of the QHE, but the proportionality of the conductance with n_e/B is clearly at odds with the observations in the QHE. We will have to break the translation invariance of our model by introducing impurities. But let us first take a more detailed look at the 'ideal' system without impurities.

3.2 Setting the scene; Some definitions

The Hamiltonian for an electron in an external potential $A^\mu = (A^0, \vec{A})$ is given by

$$H = -\frac{\hbar^2}{2m}\vec{D}^2 + qA^0, \tag{3.2.1}$$

with $\vec{D}$ the covariant derivative,

$$D_j = \partial_j + i\frac{e}{\hbar}A^j \qquad j = 1, 2 \tag{3.2.2}$$

and $-e$ the electron charge.

We will take $\vec{E} = 0$, as a nonzero electric field can be obtained simply by a change of reference frame. In this case the electrostatic potential is constant in space, since an electron feels no Coulomb force arising from impurities, other electrons or an external electric field. Without loss of generality we can set $A^0 = 0$.

The QHE is independent of the geometry of the sample, so let us choose a simple one: a disk. The most convenient choice for the vector potential $\vec{A}$ in this geometry is the so-called symmetric gauge,

$$\vec{A}(x,y) = \frac{B}{2}\begin{pmatrix} y \\ -x \end{pmatrix} = -\frac{B}{2}r\vec{e}_\phi \quad ; \quad \vec{B} = \nabla \times \vec{A} = -B\vec{e}_z \tag{3.2.3}$$

where r and ϕ are cylindrical coordinates. It will be convenient to introduce the constant ℓ, defined as

$$\ell := \sqrt{\frac{2\hbar}{eB}}. \tag{3.2.4}$$

It has the dimension of a length and is called the 'magnetic length', as it is related to the radius of the circular path of a classical electron in a field B. It allows us to define dimensionless quantities in a natural way. In the following we will use this ℓ as a unit of length.

It will also be convenient to use dimensionless complex coordinates (z,$\bar{z}$) and derivatives ∂ and $\bar{\partial}$ with respect to z and $\bar{z}$.

$$\begin{aligned} z &:= \frac{1}{\ell}(x+iy) \quad ; \quad \bar{z} := \frac{1}{\ell}(x-iy) \\ \partial &:= \frac{\ell}{2}(\partial_x - i\partial_y) \quad ; \quad \bar{\partial} := \frac{\ell}{2}(\partial_x + i\partial_y) \\ \partial z &= \bar{\partial}\bar{z} = 1 \quad ; \quad \bar{\partial} z = \partial\bar{z} = 0 \end{aligned} \tag{3.2.5}$$

Although z and $\bar{z}$ are each other's complex conjugate, they are treated as independent variables.

The Schrödinger equation is solved using the standard procedure of defining step operators such that the Hamiltonian and the angular momentum $J := -i\hbar\partial_\phi$ take the form of counting operators.

$$H = \hbar\omega_c(a^\dagger a + \frac{1}{2}) \quad ; \quad J = \hbar(b^\dagger b - a^\dagger a). \tag{3.2.6}$$

The ladder operators are given by

$$\begin{aligned} a := D_{\bar{z}} &= \frac{1}{2}z + \bar{\partial} \\ a^\dagger := -D_z &= \frac{1}{2}\bar{z} - \partial \\ b := D_z(-B) &= \frac{1}{2}\bar{z} + \partial \\ b^\dagger := -D_{\bar{z}}(-B) &= \frac{1}{2}z - \bar{\partial}. \end{aligned} \tag{3.2.7}$$

The a and $a^\dagger$ are covariant derivatives. The derivatives $D_j(-B)$ are obtained by replacing B with $-B$. The operators b and $b^\dagger$ generate translations combined with a gauge transformation corresponding to an equal translation of the origin.

The ladder operators obey standard commutation relations, and a and b commute.

$$[a, a^\dagger] = [b, b^\dagger] = 1 \quad ; \quad [a^{(\dagger)}, b^{(\dagger)}] = 0. \tag{3.2.8}$$

From the form of H and J (3.2.6) we see that

- an a-excitation carries an energy $\hbar\omega_c$ and has negative angular momentum.
- a b-excitation is 'gapless' (without energy) and has positive angular momentum.
- the angular momentum commutes with the Hamiltonian. Therefore, the eigenfunctions of H are also eigenfunctions of J.
- the Hamiltonian is infinitely degenerate with respect to the angular momentum, since H does not depend on the number of b-excitations.

In the following section we will exploit the consequences of these remarks.

3.3 Solving the Schrödinger equation; Landau levels

The eigenfunctions of H and J are obtained in the standard way. The ground state $\psi_{00}(z, \bar{z})$ is found by imposing that it is annihilated by the step-down operators a and b,

$$a\psi_{00}(z, \bar{z}) = b\psi_{00}(z, \bar{z}) = 0, \tag{3.3.1}$$

which yields

$$\psi_{00}(z, \bar{z}) = \pi^{-1/2} \exp(-\frac{1}{2}z\bar{z}). \tag{3.3.2}$$

The normalization has been chosen such that $\int d^2z\,|\psi_{00}|^2 = 1$. The other eigenstates are created by acting on ψ_{00} with $a^\dagger$ and $b^\dagger$,

$$\psi_{nj}(z,\bar{z}) = \frac{(b^\dagger)^{j+n}}{\sqrt{(j+n)!}}\frac{(a^\dagger)^n}{\sqrt{n!}}\psi_{00}(z,\bar{z}) = \sqrt{\frac{n!}{\pi(n+j)!}}z^j L_n^j(z\bar{z})\exp(-\frac{1}{2}z\bar{z}), \tag{3.3.3}$$

with n a nonnegative integer and j an integer $\geq -n$. The L_n^j are generalized Laguerre polynomials, defined as

$$L_n^j(u) := \frac{1}{n!}e^u u^{-j}\frac{d^n}{du^n}(e^{-u}u^{n+j}) = \sum_{k=0}^{n}\left(\begin{array}{c} n+j \\ n-k \end{array}\right)\frac{(-u)^k}{k!}. \tag{3.3.4}$$

The eigenfunctions are orthonormal in the sense that

$$\int d^2z\psi_{nj}^*\psi_{n'j'} = \delta_{nn'}\delta_{jj'} \tag{3.3.5}$$

and the eigenvalue equations are

$$\begin{aligned} H\psi_{nj} &= \hbar\omega_c(n+\frac{1}{2})\psi_{nj} \\ J\psi_{nj} &= \hbar j\psi_{nj}. \end{aligned} \tag{3.3.6}$$

The energy levels, labeled by n, are called *Landau levels* (LL), named after Landau, who solved the Schrödinger problem of charged particles in a constant magnetic field.

For $j >> n$ the probability density $|\psi_{nj}|^2$ has the form of $n+1$ sharp concentric rings around the origin. The outermost ring has radius r_{max},

$$r_{\max}(n,j) = \sqrt{j} + \mathcal{O}(n), \qquad j >> n. \tag{3.3.7}$$

The largest value of j that can occur on a disc of radius R is

$$j_{\max} = R^2. \tag{3.3.8}$$

Wavefunctions with higher values of j would 'fall off' the disc. Due to the finite radius of the disc, the number of states N_{LL} in a given Landau level is

$$N_{LL} = R^2. \tag{3.3.9}$$

Let us now construct a noninteracting gas of electrons from the single-electron wavefunctions ψ_{nj}. A many-fermion wavefunction is antisymmetric under the exchange of any two fermions. As our electrons are spin-polarized, the spin part of their wavefunction is symmetric under electron exchanges. Therefore, antisymmetry of the total wave function requires the position part of the wavefunction to be antisymmetric. So only one electron can occupy a state ψ_{nj}, and the maximum number of electrons that can be accomodated in a Landau level is N_{LL}.

We can define a quantity called the *filling fraction* ν as the number of filled Landau levels. If we have a total of $N_e = \ell^2 n_e \pi R^2$ electrons in our disc, then

$$\nu := \frac{N_e}{N_{LL}} = \frac{n_e h}{eB}. \tag{3.3.10}$$

Figure 3.3.1: Probability density for the state $\psi_{2,16}$

If $\nu = i + f$, $i \in N$, $f \in [0,1)$ it is implied that the lowest i Landau levels are completely filled and that the $(i+1)$'th level contains $f \cdot N_{LL}$ electrons.

The conductivity of the electron gas in a constant electric field can now be written in terms of the filling fraction ν. Equation (3.1.3) expresses the conductivity in the electron density n_e. Substitution of (3.3.10) into (3.1.3) gives

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \nu \frac{e^2}{h} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \tag{3.3.11}$$

For integer values of ν the Fermi energy lies in a big gap. In the semiclassical picture, presented in section 2.2, this gap would lead to an infinite scattering time τ_0. In that picture, we saw that we needed the scattering time to become infinite in order to obtain correspondence between theory and experiment. This shows us that there is something special about integer filling fractions.

3.4 A special case: the radial electric field

We give an explicit illustration of what happens when the electric field is turned on. For simplicity we take an electric field in the radial direction, but the results are generic. In the case of the radial field the Schrödinger equation is readily solvable, yielding wave functions that are just rescalings of the ψ_{nj} found in (3.3.3). The energies of the rescaled wave functions depend on j, which shows that the electric field has the effect of lifting the energy degeneracy. In the next section we will use this important property to gain a better understanding of impurities and the behaviour of electrons at the edge of a sample.

A radial electric field can be written conveniently in the form

$$\vec{E} = \frac{\hbar B}{2m\ell^2}\alpha(\alpha+2)\vec{r}, \tag{3.4.1}$$

where α is a real-valued parameter. The Hamiltonian H (3.2.6) is changed to $H' := H + V(|z|)$, with $V(|z|)$ a quadratic potential,

$$V(|z|) = \hbar\omega_c \cdot \frac{1}{4}\alpha(\alpha+2)|z|^2 \tag{3.4.2}$$

The orthonormal eigenfunctions of H' are obtained by a simple rescaling of the ψ_{nj}

$$\psi'_{nj}(z) = \sqrt{1+\alpha} \cdot \psi_{nj}(z\sqrt{1+\alpha}), \tag{3.4.3}$$

with eigenvalues given by

$$\begin{aligned} H'\psi'_{nj} &= \hbar\omega_c \left[n + \frac{1}{2} + \alpha(n + \frac{1}{2} + \frac{1}{2}j) \right] \psi'_{nj} \\ J\psi'_{nj} &= \hbar j \psi'_{nj}. \end{aligned} \tag{3.4.4}$$

For small enough α the change in energy is always smaller than the Landau level spacing $\hbar\omega_c$. Also, the number of states in a given Landau level, N_{LL}, does not change significantly if α is small.

The contribution to the electric current density of an electron in the state ψ'_{nj} is given by

$$\vec{J}^{(n,j)} = -e\frac{\hbar}{2m\ell^2 i}[\psi'^{*}_{nj}\vec{D}\psi'_{nj} - h.c.]. \tag{3.4.5}$$

The total current carried by a full n'th Landau level is therefore

$$\vec{J}^{(n)} = \sum_{j=-n}^{N_{LL}-n-1} \vec{J}^{(n,j)}. \tag{3.4.6}$$

In the thermodynamic limit $N_{LL} \to \infty$, explicit calculation of $\vec{J}$ for a weak electric field using (3.3.3), (3.4.3) and (3.4.5) yields the correct result for the linear response,

$$\begin{aligned} J_r^{(n)} &= 0 \\ J_\phi^{(n)} &= -\frac{e\omega_c}{2\pi\ell}\alpha|z| = -\frac{e^2}{h}E_r + \mathcal{O}(\alpha^2). \end{aligned} \tag{3.4.7}$$

Near the edge of a sample, the potential rapidly goes to infinity. The energy of electrons finding themselves near the edge will be strongly shifted upward.

3.5 Impurities; Localised versus extended states

Following the presentation of Prange in [3], we break translation invariance by introducing impurities, and show that both the occurrence of Hall plateaux and the behaviour of the longitudinal resistance in the *integer* QHE can be understood in terms of *mobility gaps* created by the impurities.

3.5.1 Mobility gaps

Numerical studies [4] have shown that no matter what kind of disorder potential is added to the Hamiltonian, the effect is always the same: The number of states remains unaltered but the states are changed in such a way that they can be divided into two kinds:

- *localised* states, confined to a small region centered on an impurity. These states do not contribute to the net current and their energy is shifted away from E_{nj} by the nearby impurities.
- *extended* states that spread through the whole system, touching all the edges of the sample and staying clear of the impurities. These are the states that carry the current. Their energy is almost unshifted, because they do not come near the impurities. The states ψ_{nj} and ψ'_{nj} found in the two previous sections are extended.

The effect of impurities on the density of states is shown in figure (3.5.1). The δ-peaks at the Landau energies are broadened and smoothened out into a continuous distribution of states. Extended states can only be found in the near vicinity of the Landau energies. The total number of states is unchanged,

$$\int_{E_p-\hbar\omega_c/2}^{E_p+\hbar\omega_c/2} N(E)dE = N_{LL}. \tag{3.5.1}$$

The difference between the two types of states is of fundamental importance. As long as the occupation of extended states does not change, the current is constant. This explains the occurrence of Hall plateaus: Figure (2.1.2), showing the resistance as a function of the filling fraction ν, can be read as follows. When the filling fraction has an integer value p, the Fermi level lies halfway between the Landau energies E_p and E_{p+1}. Changing ν means

Figure 3.5.1: The effect of impurities on the density of states $N(E)$

changing the Fermi level, but far away from the Landau energies, a change in the Fermi level leads to a change in the occupation of *localised* states only, which does not affect the current. It is said that the Fermi level lies in a *mobility gap*. Therefore, the current is independent of ν as long as the Fermi level does not come near the Landau energies.

The transition from one plateau to another takes place when ν is half-integer, because then the Fermi level crosses a Landau energy, which is where the states are extended. The Hall conductance increases with increasing number of occupied extended states, and we see $1/R_H$ rise from one plateau value to the next. The slope of this curve is proportional to the density of extended states.

The behaviour of the longitudinal resistance R_{xx} can be understood as follows: A nonzero longitudinal resistance means that there is dissipation, i.e. mobile electrons are somehow exchanging energy with the medium they are moving in. The mobile electrons are those that occupy extended states. At low temperatures, hopping between extended states with different energies is possible only for electrons with energies in the near vicinity of the Fermi level. From this we see that the longitudinal resistance is proportional to the density of extended states at the Fermi level, $N_{ext}(E_F)$. As we saw before that the slope of $1/R_H$ has the same dependence on $N_{ext}(E_F)$, we find

$$R_{xx} \propto \frac{d}{d\nu}(\frac{1}{R_H}), \tag{3.5.2}$$

which corresponds with the experiments.

In most physical problems it is only the density of states that matters. The QHE is very special in that the properties of the states are more important than the density of states.

3.5.2 Gauge argument

What remains to be understood is how it is possible that the small number of extended states at the n'th Landau level carries the same current as N_{LL} states ψ_{nj} in the absence of impurities. There is a beautiful explanation due to Laughlin [5] and Halperin [6] that is based on gauge considerations.

Figure 3.5.2: Energy of states in the Corbino disc as a function of position

Imagine a disc with a hole in it (called a Corbino disc), and impurities confined to a region well away from the edges. We take the total number of electrons such that exactly p Landau levels are filled in the impurity-free 'guard' rings. In this way we will be able to relate the behaviour of the impure region to that of the guard rings. Figure (3.5.2) shows the possible electron energies as a function of the radial coordinate r in the Corbino disc. In the pure regions all electron states have the unshifted Landau energies; In the impure regions there is an energy band; The edge states have energies going to infinity. The Fermi level lies between the (p)'th and the $(p+1)$'th impurity band. All states below the Fermi level are occupied by electrons.

The electrons cannot know whether the magnetic flux through the hole in the disc is zero or an integer multiple of the flux quantum $\Phi_0 := h/e$. What happens if we adiabatically change the flux by one flux quantum? Only extended states are affected by the flux change, because only they circle around the hole. The flux change induces an electric field satisfying

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}, \tag{3.5.3}$$

for any contour C enclosing the hole. In the guard rings the conductivity is off-diagonal, satisfying (3.3.11) with $\nu = p$, so if C lies in a guard ring then p electrons will be transferred across C. As the impure region is flanked by the guard rings, charge conservation tells us that p electrons have to enter the impure region on one side and leave it on the other side.

After the flux change all states are unaltered, since electrons do not feel a difference of one flux quantum. The only difference between the initial and final state of the system is that all extended electrons (localized ones do not notice anything) have moved one place, resulting in a net transfer of p electrons from one edge to the other. There are two possiblities: Either electrons were trapped at one side of the impure region and released at the other side, or there has been a flow of current *through the impure region*. In the first case the impure region is an insulator. We are not interested in insulators now and will not discuss this case further. The second case exactly describes the integer QHE. No matter what the filling fraction of the impure region is, as long as the Fermi level lies well away from a Landau energy, the current through the impure region will have the same magnitude as that through a pure material with integer value of ν.

This is a remarkable result. The charge transfer across the impure region shows that at least one extended state per Landau level existed during the flux change. Furthermore, even if there was only one extended state per Landau level, this was sufficient to carry the whole current.

The energy difference between the initial and final state is infinitesimal. There has been no dissipation, which means that the current was purely perpendicular to the electric field. This shows that not only the magnitude, but also the direction of the current is exactly that of the integer QHE.

3.5.3 Some different types of impurity

The generic localisation effect of impurities can be illustrated in an elegant way if we only look at impurity potentials V of the form

$$V = V_{weak} + V_{scatter} + V_{smooth}. \tag{3.5.4}$$

The weak potential satisfies

$$\max |V_{weak}(\vec{x})| =: V_{\max} << \hbar\omega_c \tag{3.5.5}$$

with no further restrictions. It comes from impurities at a distance of order ℓ from the inversion layer.

The scattering potential $V_{scatter}$ has to be zero outside some regions called scattering regions. The only condition on these scattering regions is that there has to exist a potential-free region of minimum width $d >> \ell$ percolating through the system. The scattering potential is caused by defects and charged impurities in the middle of the inversion layer.

The smooth potential V_{smooth} satisfies

$$\ell|\nabla V_{smooth}| << \hbar\omega_c. \tag{3.5.6}$$

It comes from macroscopic inhomogenities and fluctuations in the impurity density at distances larger than ℓ from the inversion layer.

The potentials that can be obtained with the prescription (3.5.4) do not cover all possible realistic impurity potentials, but they are diverse enough to be representative for the real situation. The aim of this discussion is not to give a thorough, complete treatment of impurities but to gain some insight into the mechanism of electron localisation in the QHE.

In the following we will always assume the presence of the radial electric field (3.4.1).

3.5.4 The weak impurity potential; Perturbation theory

The weak potential case can be solved with the use of perturbation theory. Let us take an impurity potential that consists of a number of delta functions,

$$V_{weak}(z) = \hbar\omega_c \sum_i \lambda_i \delta^2(z - z_i) \qquad |\lambda_i| << 1 \quad \forall i \tag{3.5.7}$$

The perturbed Hamiltonian is $H = H_0 + V_{weak}$, with H_0 given by (3.4.2). The perturbed energies and states are written as

$$\begin{aligned} E_{nj} &= E_{nj}^{(0)} + E_{nj}^{(1)} + E_{nj}^{(2)} + \dots \\ \psi_{nj} &= \psi_{nj}^{(0)} + \psi_{nj}^{(1)} + \psi_{nj}^{(2)} + \dots \end{aligned} \tag{3.5.8}$$

where the superscript $^{(n)}$, $n > 0$, denotes the n'th order correction of the unperturbed quantities $E_{nj}^{(0)}$ and $\psi_{nj}^{(0)}$ of equation (3.4.3) and (3.4.4). Solving the Schrödinger equation $H\psi_{nj} = E_{nj}\psi_{nj}$ gives to first order

$$\begin{aligned} E_{nj}^{(1)} &= \hbar\omega_c \sum_i \lambda_i |\psi_{nj}^{(0)}(z_i)|^2 \\ \psi_{nj}^{(1)} &= \sum_{\substack{m,k \\ (m,k)\neq(n,j)}} \frac{\int d^2z \psi_{mk}^{(0)*} V_{weak} \psi_{nj}^{(0)}}{E_{nj}^{(0)} - E_{mk}^{(0)}} \psi_{mk}^{(0)} \\ &= \sum_i \lambda_i \sum_{\substack{m,k \\ (m,k)\neq(n,j)}} \frac{\psi_{mk}^{(0)*}(z_i)\psi_{nj}^{(0)}(z_i)}{(1+\alpha)(n-m)+\alpha(j-k)/2} \psi_{mk}^{(0)}. \end{aligned} \tag{3.5.9}$$

The energy of a state $\psi_{nj}^{(0)}$ is shifted only if $\psi_{nj}^{(0)}$ 'touches' a delta potential. The shift is much smaller than the Landau level spacing $\hbar\omega_c$. The state is changed in such a way that it gets trapped by the delta potential: it becomes localised. States with $|\psi(z_i)| << 1$ are hardly affected at all. They stay extended.

Using the gauge argument, we find the correct value for the current.

3.5.5 The scattering potential

The movement of an electron through a system that contains huge potentials in regions with sharp boundaries can be seen as a scattering problem. The temperature is very low, so we will only look at elastic scattering. Imagine a wave packet of precisely defined energy moving in the potential-free region and impacting on a scatterer. After the scattering, the wave packet will find itself in the potential-free region again, in a state with the same energy as before, but moving away from the scatterer. In the case of the radial electric field, every possible energy is determined by a unique pair (n, j). If the ingoing state is ψ_{nj} then the outgoing state is also ψ_{nj}, because no other state has the same energy. The outgoing state can differ from the ingoing one by at most a phase shift $\delta(j)$. (δ does depend on n, but much weaker than on j, since j determines the radial position of the wave packet.)

$$\psi_{nj}^{out} = e^{i\delta(j)}\psi_{nj}^{in}. \tag{3.5.10}$$

We can now impose periodicity of the scattered wave function. After one complete revolution around the disc, the wave function has aquired two phase shifts: One due to the complete rotation and one due to the scattering at the site of the impurity. The latter is $\delta(j)$, the former is found using $\psi_{nj} \propto z^j$. This gives the following condition

$$\begin{aligned} e^{i\delta(j)} e^{ij(\phi+2\pi)} &= e^{ij\phi} \\ \Longleftrightarrow \quad 2\pi j + \delta(j) &= 2\pi p, \quad p \in Z. \end{aligned} \tag{3.5.11}$$

The scattered wave functions can be labeled by the integer p. For each p there is one j (generally non-integer!) that satisfies (3.5.11). In the absence of impurities, the p and j labeling coincide.

The j-dependence of δ is in fact an r-dependence, since a state ψ_{nj} occupies a ring of radius $r \approx \sqrt{j}$. If the circle with radius $r = \sqrt{j}$ does not touch the scatterer, then the wave function ψ_{nj} is unaltered. In every r-interval that is free of scatterers, $\delta(j)$ is a constant

Figure 3.5.3: Electrons moving around the scattering regions

integral multiple of 2π. As j increases from the 'bottom' end of the scatterer to the 'top', $\delta(j)$ changes by an integer times 2π. If this change is $-2\pi M$, with M a nonnegative integer, there will be M fewer solutions of (3.5.11) than if there were no scatterers at all. The interpretation is that the missing solutions are localised states, bound to the scatterer, as the total number of states is conserved. Negative values of M do not make sense; they would increase the number of states.

The flow of electric current can be visualised as in figure (3.5.3). The electrons skirt the edges of the scattering regions. At these edges, the electric field $-\nabla V_{scatter}$ is very strong, causing an increase in the electrons' speed. This speeding up can also be seen as follows: During the time interval t needed to travel around a scatterer, a wave function with energy E picks up a phase $\exp i[\delta(E) - Et/\hbar]$. The energy can be used as a radial coordinate, since $E \propto j \propto \sqrt{r}$. We can make a Taylor expansion in E of the phase shift around E_0, the energy at the bottom of a scatterer. Using $\exp[i\delta(E_0)] = 1$ we find

$$e^{i[\delta(E)-Et/\hbar]} \approx e^{-i\frac{E}{\hbar}[t-\hbar\frac{E-E_0}{E}\frac{d\delta}{dE}]} =: e^{-i\frac{E}{\hbar}[t-t_{delay}]}. \tag{3.5.12}$$

As the time delay $t_{delay} \propto d\delta/dE$ is negative, we actually have a time advance. This explains the mechanism of how a reduced number of extended states can carry the same current as a lot of extended states in the absence of scattering impurities.

Again the gauge argument is used to find the correct value for the current.

It is not surprising that the presence of scattering potentials does not destroy the QHE. Introducing these strong potentials is similar to cutting holes in the system, but the QHE is independent of the geometry of the sample.

3.5.6 The smooth impurity potential

Since the smooth impurity potential varies very slowly, we can think of the system as a patchwork of regions where the electric field is constant. Every such region exhibits the quantum Hall effect. The electrons move along equipotential lines. Closed equipotential lines lead to localised states, while states following equipotential lines that reach throughout the system are extended.

For zero external electric field and completely random impurity potential there is only one unique energy of extended states: If a state is extended in the x-direction, it has to be extended in the y-direction as well, since the impurity potential has no preferred direction. Equipotential lines of different energies cannot cross, so there is only one energy

Figure 3.5.4: Equipotential lines for a smooth impurity potential

for extended states. This energy lies halfway between the average top of the potential 'hills' and the average bottom of the 'valleys', since closed equipotential lines walk around hills as well as valleys. The extended state energy is approximately the Landau energy. An external electric field introduces a preferred direction and can create extra extended equipotential lines at new energies.

It is easy to see that the smooth potential does not affect the macroscopic flow of current.

In figure (3.5.4), the voltage drop between S' and S is given by

$$V_{SS'} = V_{S'A} + V_{AS}, \tag{3.5.13}$$

since $V_{AA} = 0$. In both regions SA and AS' current flows through the line SS' The region AA gives no net contribution to the current, as the equipotential lines are closed. Together the transverse currents $I_H^{(SA)}$ and $I_H^{(AS')}$ yield the total macroscopic current in the azimuthal direction. Using (3.4.7) for integer filling fraction and (3.5.13) we find

$$I_\phi = I_H^{(SA)} + I_H^{(AS')} = \sigma_H(V_{AS} + V_{S'A}) = \sigma_H V_{SS'}. \tag{3.5.14}$$

It is as if the impurity does not even exist.

Chapter 4

The fractional effect

...en de metingen waren niet van de werkelijkheid te onderscheiden!

Vincent Icke

In chapter 3 we saw that the integer QHE is well understood in terms of a two-dimensional gas of noninteracting electrons that occupy localised or extended states. However, this model does not predict the fractional effect. In the noninteracting gas, there is no mechanism that could cause mobility gaps to occur around fractional values of ν. The fractional QHE is realised only in very pure samples. Since the impurity potential in these samples is weak and long-ranged, the interelectron repulsion can no longer be neglected.

In this chapter we return to ideal samples without impurities. We introduce the Coulomb interaction and present the famous Laughlin wave functions, found by Laughlin in 1983 [8], describing an *incompressible quantum fluid* at filling fraction $\nu = \frac{1}{q}$, with q an odd integer. Here we encounter the important aspect of incompressibility for the first time. The Laughlin wave functions are a very good approximation to the exact ground state for a large class of repulsive potentials.

Then we investigate the nature of the quasiparticle/hole excitations and show that they are anyons. At that point we give a short review on anyons, explaining why anyons are at all possible in two dimensions and how they can be described in terms of charge-flux composites. (For a thorough treatise on anyons see [13].) It is natural to assume that impurities will have the generic effect of localisation also for the quasiparticle/hole excitations of the electron ground state, thereby causing the occurrence of plateaus at $\nu = 1/q$.

Finally we present two ways of explaining the occurrence of the QHE at all the observed fractions: the hierarchy of Haldane [9] and Halperin [10], based on the anyonic excitations of the Laughlin states, and Jain's hierarchy [14], based on the integer QHE and the idea of composite fermions.

4.1 The Laughlin wave functions

In order to be able to make a comparison between the interacting and non-interacting case, we will first take a look at the N-particle wave functions of the non-interacting system. The N-particle Hamiltonian for a gas of N electrons in a uniform magnetic field in the absence of impurities is given by

$$H = \hbar\omega_c \sum_{k=1}^{N} (a_k^\dagger a_k + \frac{1}{2}). \tag{4.1.1}$$

The total angular momentum operator is

$$J = \frac{\hbar}{i} \sum_{k=1}^{N} \frac{\partial}{\partial \phi_k} = \hbar \sum_{k=1}^{N} (b_k^\dagger b_k - a_k^\dagger a_k) \tag{4.1.2}$$

The differential operators $a_k^{(\dagger)}$ and $b_k^{(\dagger)}$ are the ladder operators as defined in (3.2.7), referring to the k'th electron.

The eigenfunctions of H and J are simply found by antisymmetrisation of a product of N of the single electron wave functions ψ_{nj} defined in (3.3.3). The antisymmetrised product is a Slater determinant. A special example is the wave function for the completely filled lowest Landau level (LLL), where the Slater determinant takes the form of a Vandermonde determinant.

$$\begin{aligned}
\psi_{LLL}(z_1, \cdots, z_N) &= \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{00}(z_1) & \cdots & \psi_{00}(z_N) \\ \vdots & \ddots & \vdots \\ \psi_{0,N-1}(z_1) & \cdots & \psi_{0,N-1}(z_N) \end{vmatrix} \\
&= \left[\pi^N N! \prod_{j=0}^{N-1} j! \right]^{-1/2} \exp(-\frac{1}{2} \sum_{k=1}^{N} |z_k|^2) \prod_{i<j}^{N} (z_i - z_j),
\end{aligned} \tag{4.1.3}$$

with $N = N_{LL}$, the number of states in a Landau level. This wave function is clearly antisymmetric under an exchange of any two electrons. The eigenvalues are

$$\begin{aligned}
E &= N \cdot E_0 = \frac{N}{2} \hbar\omega_c \\
J &= \hbar \sum_{j=0}^{N-1} j = \frac{\hbar}{2} N(N-1).
\end{aligned} \tag{4.1.4}$$

We now introduce the Coulomb interactions into our model by adding two extra terms to the Hamiltonian (4.1.1); the two-particle repulsion and an attraction to a neutralising positive background potential.

$$H_{int} = \sum_{k=1}^{N} \left[\hbar\omega_c (a_k^\dagger a_k + \frac{1}{2}) - \rho_0 \int d^2 z \frac{e}{4\pi\epsilon |z - z_k|} \right] + \sum_{j<k}^{N} \frac{e^2}{4\pi\epsilon |z_j - z_k|}, \tag{4.1.5}$$

where ρ_0 is the positive charge density and ϵ is the dielectric constant in the sample (both in suitable units).

From the study of liquid Helium it is known that a good approximation to the ground state of a system in which particles want to avoid one another is given by a Slater determinant multiplied by a factor

$$\prod_{j<k}^{N} f(z_j - z_k), \tag{4.1.6}$$

where f is an as yet unknown function that goes to zero when its argument goes to zero. This behaviour of f reflects the electron-electron repulsion. The resulting form of the wave function is called the Jastrow form. An approximate ground state can be found by varying f until the energy of the wave function is minimized.

Laughlin [8] realised that when one tries to find an approximate ground state of the Jastrow form for an interacting quantum Hall system in a very strong magnetic field, there is almost no freedom left in the choice of f. The Slater determinant is given by ψ_{LLL} (4.1.3), since the magnetic field is so strong that the filling fraction $\nu \leq 1$. For a rotationally symmetric sample in a very strong magnetic field, f has to satisfy the following powerful restrictions:

- The N-electron wave function is totally antisymmetric. Therefore, f has to be an even function.
- When the magnetic field is strong, the first term in the Hamiltonian (4.1.5) is the most important one (as $\omega_c \propto B$). As a consequence, the system will almost behave as a gas of electrons in lowest Landau level states. It is then a good approximation to require that the wave function consists of single-electron wave functions in the LLL. This is accomplished only if $f(z)$ is an analytic function of z.
- The wave function is an angular momentum eigenstate.

The only function f satisfying these conditions is $f(z) = z^{2p}$, with p a nonnegative integer. The N-electron wave function then becomes

$$\psi_p(z_1, \cdots, z_N) \propto \exp(-\frac{1}{2}\sum_{k=1}^{N} |z_k|^2) \prod_{i<j}^{N} (z_i - z_j)^{2p+1}. \tag{4.1.7}$$

The wave functions ψ_p are called Laughlin wave functions. They are totally antisymmetric, as an exchange of any two electrons yields a factor $(-1)^{2p+1} = -1$. The angular momentum eigenvalue of ψ_p is

$$J_p = (2p+1) \cdot \frac{\hbar}{2} N(N-1). \tag{4.1.8}$$

The value of ρ_0 in (4.1.5) determines for which p the energy of ψ_p is lowest.

One may fear that the assumption of the Jastrow form is wrong or gives incomplete results. However, the overlap of the Laughlin wave functions (4.1.7) with numerically obtained ground states is extremely high for various repulsive potentials, such as $1/|z|$, $-\ln|z|$ and $\exp(-|z|^2/2)$. This shows that the single fact of repulsion is more important than the exact form of the repulsive force. Furthermore, no other ground state has ever been found either numerically or analytically that has has an energy as low as the Laughlin functions.

For $p = 0$ we regain the wave function ψ_{LLL} of a completely filled lowest Landau level, but what do the Laughlin functions with $p \neq 0$ describe? A nice way of finding out is to search for a configuration of positions $\{ z_1, \cdots, z_N \}$ that maximizes the probability density $|\psi_p|^2(z_1, \cdots, z_N)$ of encountering that configuration. To this end we write the probability as

$$|\psi_p|^2(z_1, \cdots, z_N) = e^{-V(z_1, \cdots, z_N)} \tag{4.1.9}$$

with V some kind of effective potential energy that represents the electron configuration. This approach is called the plasma analogy. Using (4.1.7) we find

$$V(z_1, \cdots, z_N) = \frac{4\pi}{2p+1} \left[-\frac{(2p+1)^2}{2\pi} \sum_{i<j}^{N} \ln|z_i - z_j| + \frac{2p+1}{4\pi} \sum_{k=1}^{N} |z_k|^2 \right]. \tag{4.1.10}$$

The most probable configuration is the one that minimizes this potential energy. Remarkably enough, the part between brackets can be interpreted as the potential energy of a two-dimensional plasma: N particles of 'charge' $2p+1$ with a logarithmic interaction that are attracted to the origin by a background charge of uniform 'charge' density $1/\pi$. [Recall that in two dimensions the electric field of a point charge is given by $E_r(r) = Q/(2\pi r)$ and that the field in a medium of constant density ρ is $(\rho\pi r^2)/(2\pi r) = (\rho/2)r$. Integration gives the potentials $-Q(2\pi)^{-1}\ln(r)$ and $(\rho/4)r^2$ respectively.] The plasma's energy is lowest when it is neutral, i.e. when the 'charge' density of the configuration $\{ z_1, \cdots, z_N \}$ is a uniform $1/\pi\ell^2$. Since the 'charge' of the electrons is $2p+1$, this corresponds to an electron density $n_e = \frac{1}{(2p+1)\pi\ell^2}$. Any configuration with a different density has a much higher energy, and the exponentiation (4.1.9) rapidly sends the probability of encountering such a configuration to zero. Consequently, the probability of measuring an electron density $n_e = \frac{1}{(2p+1)\pi\ell^2}$ is overwhelming. Using the definitions of the filling fraction ν (3.3.10) and the magnetic length ℓ (3.2.4), we find that a Laughlin state ψ_p is the ground state of an interacting electron system at filling fraction

$$\nu_p = \frac{1}{2p+1}. \tag{4.1.11}$$

The translational invariance argument (3.1.3) tells us that switching on an electric field will cause a Hall current, such that the conductivity is given by

$$\sigma_H = \frac{1}{2p+1}\frac{e^2}{h}. \tag{4.1.12}$$

4.1.1 Excitations with fractional charge and statistics

Numerical studies have shown that every Laughlin state lies at a deep energy minimum; States with filling fractions other than ν_p have much higher energies. This energy difference is caused by the electron-electron repulsion. When the Coulomb interaction in (4.1.5) is slowly turned off, the depth of the minima gradually decreases. When there is no Coulomb interaction at all, we reobtain the non-interacting picture, in which there is no energy minimum at all at the fractions ν_p.

In cases like the FQHE, when there is an energy gap for density fluctuations, it is said that the system is an *incompressible quantum fluid.* The word 'fluid' is used because the interacting electron system does not behave like a gas or like a crystal with fixed lattice positions. The aspect of incompressibility is of fundamental importance for understanding the FQHE. The algebraic treatment of incompressibility will be discussed in chapter 6.

By varying the filling fraction ν away from ν_p we create excitations of the ground state ψ_p. In chapter 3 we chose to view a variation of ν as a variation of the electron number at constant magnetic field strength. Now we will view it as changing B while N is kept constant. Imagine that B is changed only at one point z_0 in the sample. It is done adiabatically, so the wave function is an eigenstate of the changing Hamiltonian at any moment in time. When the flux through z_0 is exactly one flux quantum, $\Phi = \pm h/e$, the Hamiltonian regains its original form. We are left with an excited state of the original Hamiltonian. Just as was the case in the gauge argument (see section 3.5.2), the induced electric field causes a current perpendicular to the electric field, but now of magnitude $|j| = \nu_p(e^2/h)E$. The electric field circles around the point z_0 and the induced current flows toward z_0 or away from it. After the flux change, there is a charge excess or deficit q_p

Figure 4.1.1: Exchange of two charge/flux-composites

at z_0,

$$q_p = \nu_p e \cdot sign(\Phi) = \frac{e}{2p+1} sign(\Phi). \tag{4.1.13}$$

This kind of excitation is called a *quasiparticle* or a *quasihole*. The prefix 'quasi' is used because the excitation is not a true particle, but a collective behaviour of many particles. Addition of more flux quanta at other positions creates more quasiparticles/holes. In the following, we will refer to quasiparticles and quasiholes alike only by the name quasiparticle, and often the abbreviation QP will be used.

If the density of QPs is low, there is no Coulomb repulsion between the QPs, as they are far apart. Impurities will have the same effect on these noninteracting QPs in the fractional QHE as on electrons in the integer QHE. The localisation mechanisms described in chapter 3 will work in the same way. Consequently, the current will be independent of the filling fraction in an interval around $\nu = \nu_p$. This explains the occurrence of plateaus at these fillings. For high QP densities there are not enough impurities to localise all QPs, so that some of them contribute to the current. This situation corresponds to the transition regions between plateaus.

For $p = 0$, the completely filled lowest Landau level, an excitation behaves like an ordinary electron or electron hole. For other values of p, the excitations have fractional charges 1/3, 1/5, ...

Not only do they have fractional charges, the quasiparticles also obey fractional statistics. This can be heuristically shown as follows: An exchange of two QPs essentially consists of a rotation over 180 degrees of one QP around the other (see figure 4.1.1). During this rotation they 'feel' each other's flux tube as they move through the vector potential corresponding to the other's flux. This causes a phase shift of the two-QP wave function. The total phase shift $\exp(i\theta)$ due to the QP fluxes during the exchange is half that of one QP making a complete rotation around the other,

$$\frac{\theta}{\pi} = \frac{1}{2\pi}\frac{q_p}{\hbar}\oint_C \vec{A}_{QP} \cdot d\vec{l} = \frac{q_p}{h}\Phi = \frac{1}{2p+1} \tag{4.1.14}$$

where C is a contour around one of the QPs. Unless $p = 0$, the quasiparticles do not have the exchange properties of fermions or bosons; The phase factor $\exp(i\theta)$ is not ± 1.

4.2 Anyons

> Rincewind set off, wondering how you attracted onions. After all, he thought, although you see them hanging in ropes on market stalls, they probably don't grow like that, perhaps peasants or whatever use onion hounds or something, or sing songs to attract onions.
>
> Terry Pratchett, *The Light Fantastic*

The behaviour of identical particles under an exchange is called 'statistics'. We are all familiar with bosons and fermions; The multiparticle wave functions aquire a phase factor of $+1$ and -1 respectively when two identical particles are exchanged. The corresponding second quantized field operators obey commutation / anticommutation relations.

In three spatial dimensions, bosonic and fermionic statistics are the only possibilities. An investigation of the two-particle configuration space will show us why this is the case. (We only have to look at the two-particle problem, since a many-particle exchange is just a combination of two-particle exchanges.)

The configuration space of two *indistinguishable* particles in three dimensions is essentially given by the projective space $\mathcal{P}_2$, the sphere with opposite points identified. (One particle is in the origin, the other moves around it. Their relative distance is unimportant and therefore considered constant. An exchange consists of a rotation over 180 degrees of one particle around the other, combined with a translation. Therefore, opposite points on the sphere have to be identified, as they represent identical configurations.) An exchange is given by a path from one point in $\mathcal{P}_2$ to the opposite point. All exchange paths in $\mathcal{P}_2$ are equivalent, since they can be mapped into each other by continuous transformations. It is not possible to continuously transform an exchange path into a path that does not represent an exchange.

For a one-component wave function ψ, the exchange of indistinguishable particles can result at most in a phase shift σ, since the probability density $|\psi|^2$ is not affected. We define

$$\sigma = e^{i\theta} = e^{i2\pi s}, \tag{4.2.1}$$

where θ and s are real numbers called the *statistical angle* and the *statistics* respectively. The phase shift does not depend on the way the exchange is made, since all exchange paths are equivalent. Two consecutive exchanges result in a closed loop from a starting configuration back to itself. The loop can be contracted into a point, which means that two exchanges have the same effect as no exchange at all. This imposes serious restrictions on the possible values of s,

$$\sigma^2 = 1 \implies s \in \{0, \frac{1}{2}\} \pmod 1. \tag{4.2.2}$$

The only possibilities are bosonic (integer s) or fermionic (halfinteger s) statistics.

In the two-dimensional plane the situation is completely different. The configuration space of two indistinguishable particles is now given by a circle with opposite points identified. There are many inequivalent ways to perform an exchange, differing in the amount of times that the particles go around each other and in which direction this happens. Thus, a path can be labeled with the integer corresponding with the winding number m. The phase shift of a many-particle wave function due to an exchange with winding number m is given by

$$\sigma = e^{im2\pi s}. \tag{4.2.3}$$

Figure 4.2.1: A braiding of three particles

There are no restrictions on s, since consecutive exchanges do not result in contractible loops in configuration space. Therefore, the statistics of two-dimensional particles could be *any*thing, which is why such particles are called *anyons*. Of course, we live in a three-dimensional world and all particles in this world are fermions or bosons. However, there are physical systems which are effectively two- dimensional. In these systems quasiparticle excitations can occur, which, being purely two-dimensional, *can* have anyonic statistics. This is the case in the Quantum Hall Effect.

Exchanges of many particles are somewhat more complicated for anyons than for fermions and bosons. In the latter case, the group that corresponds to N-particle exchanges is the permutation group S_N, while for anyons it is the braid group B_N. (Imagine that every anyon is connected to a line perpendicular to the plane. When particles are exchanged the lines get braided. See figure 4.2.1) The difference is caused by the fact that the winding numbers are important for anyons but not for fermions and bosons.

4.2.1 Topological interactions

In the path integral formulation of N-particle quantum mechanics, the amplitude K for a system evolving from a configuration q at time t back to the same configuration q at t' is given by a functional integral over all paths $q(\tau)$ that lead from q back to q. For fermions, each path also has to be weighed with a factor $(-1)^P$, where P is the number of exchanges that occur in the path (or the parity of the permutation). This gives

$$K_B(q;t,t') = \int_q^q \mathcal{D}q' \; e^{\frac{i}{\hbar}S[q']} \tag{4.2.4}$$

for bosons and

$$K_F(q;t,t') = \sum_{\alpha\in S_N} (-1)^{P_\alpha} \int_q^q \mathcal{D}q'_\alpha \; e^{\frac{i}{\hbar}S[q'_\alpha]} \tag{4.2.5}$$

for fermions. The transition amplitude K is also called the kernel.

The generalization to anyons is easy. We have to sum over all braidings instead of permutations, and the weight factor of a braiding $\alpha \in B_N$ is $\exp(i2\pi s W_\alpha)$, where W_α is the number of counterclockwise exchanges minus the number of clockwise ones that occur in α, i.e. the total winding number of α. The resulting amplitude is

$$K_s(q;t,t') = \sum_{\alpha \in B_N} e^{i2s\pi W_\alpha} \int_q^q \mathcal{D}q'_\alpha e^{\frac{i}{\hbar}\int_t^{t'} L d\tau}. \tag{4.2.6}$$

The total winding number W_α can be written as

$$\pi W_\alpha = \sum_{I<J} \int_t^{t'} d\tau \frac{d}{d\tau} \Phi_{IJ}(q_\alpha(\tau)) \tag{4.2.7}$$

where $\Phi_{IJ}(q_\alpha(\tau))$ is the winding angle of particle I with respect to particle J at time τ, given a path q_α. The time derivative $\frac{d}{dt}\Phi_{IJ}$ is given by

$$\begin{aligned} \frac{d}{dt}\Phi_{IJ} &:= \frac{d}{dt}\Phi(\vec{x}_I - \vec{x}_J) \\ &= \frac{d}{dt}\arctan\left(\frac{y_I - y_J}{x_I - x_J}\right). \end{aligned} \tag{4.2.8}$$

Insertion of (4.2.7) into (4.2.6) gives two interesting ways to write the anyonic kernel $K_s(q;t,t')$, namely

$$K_s(q;t,t') = \int_q^q \mathcal{D}q e^{\frac{i}{\hbar}\int_t^{t'} d\tau \left[L + \hbar 2s \sum_{I<J} \frac{d}{d\tau}\Phi_{IJ}(q(\tau))\right]} \tag{4.2.9}$$

and

$$K_s(q;t,t') = \sum_{\alpha \in B_N} (-1)^{W_\alpha} \int_q^q \mathcal{D}q_\alpha e^{\frac{i}{\hbar}\int_t^{t'} d\tau \left[L + \hbar(2s-1) \sum_{I<J} \frac{d}{d\tau}\Phi_{IJ}(q_\alpha(\tau))\right]} \tag{4.2.10}$$

Comparing this to (4.2.4) and (4.2.5), and noting that the weight factor $(-1)^{W_\alpha}$ is actually the same as $(-1)^{P_\alpha}$, we see that a particle with statistics s can be described as a boson or a fermion with a modified Lagrangian. The effective bosonic Lagrangian is given by

$$L_B = L + 2\hbar s \sum_{I<J} \frac{d}{d\tau}\Phi_{IJ}, \tag{4.2.11}$$

and the fermionic one by

$$L_F = L + \hbar(2s-1) \sum_{I<J} \frac{d}{d\tau}\Phi_{IJ}. \tag{4.2.12}$$

This means that anyons can be treated as ordinary bosons or fermions that have some kind of extra interaction. This interaction does not change their equations of motion, since the extra terms in L_B and L_F are total derivatives, but depends only on the topology of the configuration space A term of this kind is therefore called a 'topological interaction term'.

There is a very nice physical realisation of the statistical interaction that uses a gauge field and the Aharonov-Bohm effect. Using $\nabla_J \Phi_{IJ} = \nabla_J \Phi_{JI}$ we can write $\frac{d}{dt}\Phi_{IJ}$ as

$$\frac{d}{dt}\Phi_{IJ} = \vec{v}_I \cdot \nabla_I \Phi_{IJ} + \vec{v}_J \cdot \nabla_J \Phi_{JI} \tag{4.2.13}$$

so that

$$\sum_{I<J} \frac{d}{dt}\Phi_{IJ} = \sum_{I=1}^{N} \vec{v}_I \cdot \nabla_I \sum_{J \neq I} \Phi_{IJ}. \tag{4.2.14}$$

The statistical interaction term can now be written in the form of a coupling with a vector potential A^μ

$$L_{int} = \sum_{I=1}^{N} q(\vec{v}_I \cdot \vec{A}_I - A_I^0) \tag{4.2.15}$$

where q is the coupling constant and A_I^μ is the gauge field at the position of the I'th particle. This is an important observation which allows for an alternative description of anyonic statistics using well-known gauge theory techniques.

For bosons we can choose

$$A^0 = 0 \tag{4.2.16}$$

$$\vec{A}_I(\vec{x}_1, \cdots, \vec{x}_N) = 2\frac{\hbar s}{q} \nabla_I \sum_{J \neq I} \Phi_{IJ}, \tag{4.2.17}$$

and the same definition for fermions, except that s becomes $s - 1/2$. From now on I will only discuss the addition of statistics to bosons, since $s \to s - 1/2$ is the only difference between the fermionic and the bosonic case.

In an arbitrary point $\vec{x} \notin \{\vec{x}_K\}$ we have

$$\vec{A}(\vec{x}) = 2\frac{\hbar s}{q} \nabla \sum_{K=1}^{N} \Phi(\vec{x} - \vec{x}_K). \tag{4.2.18}$$

$\vec{A}$ looks like a pure gauge, but we have to be careful because $\Phi(\vec{x} - \vec{x}_K)$ is multi-valued and completely undefined at the locations $\vec{x} = \vec{x}_K$. Explicit calculation of the derivative in (4.2.18) gives

$$A^i(\vec{x}) = -2\frac{\hbar s}{q} \epsilon^{ij} \sum_{K=1}^{N} \frac{x^j - x_K^j}{|\vec{x} - \vec{x}_K|^2} \tag{4.2.19}$$

which can also be written as

$$A^i(\vec{x}) = -2\frac{\hbar s}{q} \epsilon^{ij} \partial_j \sum_{K=1}^{N} \ln|\vec{x} - \vec{x}_K| = -4\pi \frac{\hbar s}{q} \epsilon^{ij} \partial_j \sum_{K=1}^{N} G(\vec{x} - \vec{x}_K) \tag{4.2.20}$$

where G is the Green's function in two dimensions. From (4.2.19) we see that $\vec{A}$ goes to infinity at $\vec{x} = \vec{x}_I$.

In analogy with electrodynamics, we can define a 'magnetic' field B as

$$B(\vec{x}) := \partial_1 A^2 - \partial_2 A^1 = -2s\frac{h}{q} \sum_{K=1}^{N} \delta(\vec{x} - \vec{x}_K). \tag{4.2.21}$$

We have a 'magnetic' flux $-2s\frac{h}{q}$ at the location of every particle. Even though the hard-core particles cannot directly feel each other's fluxes by actually moving through them, they do interact with the fluxes *indirectly* by experiencing phase shifts due to the vector potential. This is nothing but the Aharonov-Bohm effect.

4.2.2 Anyonic vs. bosonic wavefunctions

If we describe anyons as bosons with an extra interaction, then the N-particle wave functions are of course bosonic,

$$\phi(\vec{x}_1, \cdots, \vec{x}_J, \cdots, \vec{x}_I, \cdots, \vec{x}_N) = \phi(\vec{x}_1, \cdots, \vec{x}_I, \cdots, \vec{x}_J, \cdots, \vec{x}_N) \tag{4.2.22}$$

Since the vector potential is a total derivative (4.2.17), we can perform a singular gauge transformation such that $\vec{A}$ becomes zero. The wave function transforms as

$$\phi \to \tilde{\phi} := \exp\left(i2s \sum_{I<J} \Phi_{IJ}\right)\phi \tag{4.2.23}$$

The price we pay is that the anyonic wavefunction $\tilde{\phi}$ is multi-valued, since Φ is not uniquely defined. We should view $\tilde{\phi}$ not as a function, but as a section of a U_1 bundle over the plane with the particle locations taken out. Moreover, the anyonic wave function $\tilde{\phi}$ satisfies different boundary conditions than ϕ. A counterclockwise exchange of particle I around particle J gives

$$\tilde{\phi}(\vec{x}_1, \cdots, \vec{x}_J, \cdots, \vec{x}_I, \cdots, \vec{x}_N) = e^{i2\pi s}\tilde{\phi}(\vec{x}_1, \cdots, \vec{x}_I, \cdots, \vec{x}_J, \cdots, \vec{x}_N) \tag{4.2.24}$$

This shows that we can trade a complicated interaction term for complicated boundary conditions on the wave functions. Another way of obtaining this result is to look at the path integral formulation. The time evolution of a wavefunction for N bosons with a statistical interaction is given by (see (4.2.6) and (4.2.7))

$$\phi(q', t') = \int dq < q', t'|q, t >< q, t|\phi >= \int dq K_s(q', t'; q, t)\phi(q, t) \tag{4.2.25}$$

where the amplitude K_s is

$$K_s(q', t'; q, t) = \sum_{\alpha \in B_N} e^{i2s\sum\limits_{I<J}[\Phi_{IJ}(t')-\Phi_{IJ}(t)]_\alpha} \int_q^{q'} \mathcal{D}Q_\alpha e^{\frac{i}{\hbar}\int_t^{t'} d\tau L(Q_\alpha, \dot{Q}_\alpha)} \tag{4.2.26}$$

The path $q \to q'$ does not have to be closed. Even so, every path can be labeled by an element of the braid group, counting how many times a *complete* winding has been made. The expression $[\Phi_{IJ}(t') - \Phi_{IJ}(t)]_\alpha$ is the generalization of the total winding number W_α for closed paths. An element of the braid group B_N can be represented by a set of winding numbers $\{n_{IJ}\}_{I<J}$, one number for each pair of particles. If we define Φ_{IJ} to lie in an interval of size 2π, for example $(-\pi, \pi]$, then we can write

$$[\Phi_{IJ}(t') - \Phi_{IJ}(t)]_\alpha = \Phi_{IJ}(t') - \Phi_{IJ}(t) + 2\pi[n_{IJ}]_\alpha \tag{4.2.27}$$

with Φ single-valued, so that

$$K_s(q', t'; q, t) = e^{i2s\sum\limits_{I<J}[\Phi_{IJ}(t')-\Phi_{IJ}(t)]} \sum_{\{n_{IJ}\}_{I<J}} e^{i4\pi s\sum\limits_{I<J} n_{IJ}} K_0^{\{n_{IJ}\}}(q', t'; q, t) \tag{4.2.28}$$

where $K_0^{\{n_{IJ}\}}(q', t'; q, t)$ is the free transition amplitude given the set of winding numbers $\{n_{IJ}\}$. Equation (4.2.25) can now be written as

$$e^{i2s\sum\limits_{I<J}\Phi_{IJ}(t')}\phi(q', t') = \sum_{\{n_{IJ}\}} \int dq K_0^{\{n_{IJ}\}}(q', t'; q, t) e^{i2s\sum\limits_{I<J}[\Phi_{IJ}(t)-2\pi n_{IJ}]}\phi(q, t) \tag{4.2.29}$$

If we now allow Φ to be multi-valued again, then $2\pi \sum n_{IJ}$ gets absorbed into Φ_{IJ} and we get, using definition (4.2.23),

$$\tilde{\phi}(q', t') = \int dq K_0(q', t'; q, t) \tilde{\phi}(q, t) \tag{4.2.30}$$

This tells us that $\tilde{\phi}$ indeed evolves with the free kernel K_0.

4.3 Hierarchy schemes

We will now present two models that use the idea of fractional statistics in order to obtain an explanation for the occurrence of plateaus at fractional values of ν. They are not explicitly formulated in terms of the statistical interaction (4.2.15) with the gauge field A^μ, but they do exploit the generalised statistics s resulting from charge-flux binding.

4.3.1 The Haldane-Halperin hierarchy of fractions

Haldane [9] and Halperin [10] proposed that an energy minimum is reached when the wave function of a collection of quasiparticles has a Laughlin-like form. One then has a quantum fluid of electrons *and* a quantum fluid of quasiparticles. What is more, by changing the density a little new excitations can be created from the quasiparticle quantum fluid. New energy minima occur when these new excitations condense into Laughlin-like states, resulting in a third quantum fluid, ... and so on. In this way, energy minima are found for many fractional filling fractions, among which the observed FQHE-values.

Both quasiparticle and quasihole excitations of the quantum liquid at $\nu = 1/m$ (with m an odd integer) have statistics $1/(2m)$. The ground state of quasiholes with statistics $1/(2m)$ is given by

$$\psi_m^+ \propto \exp(-\frac{1}{2m} \sum_k |z_k|^2) \prod_{i<j} (z_i - z_j)^{2k + \frac{1}{m}} \tag{4.3.1}$$

with k a positive integer. This form is very similar to the original ground state (4.1.7). The factor m in the exponent appears because $|z|^2$ is scaled by a factor $\ell^{-2} = eB/(2\hbar)$ and the charge of the quasihole is e/m. An exchange of two quasiholes results in a phase factor $\exp(i\frac{\pi}{m})$, which is exactly what one requires of objects with statistics $s = 1/(2m)$. (Cf. the exchange properties of $\tilde{\phi}$ (4.2.24).) The integer k does not affect this exchange property.

The quasiparticle wave function is more complicated. Laughlin tried the following ansatz

$$\psi_m^- \propto \exp(-\frac{1}{2m} \sum_k |z_k|^2) \prod_{i<j} (\bar{z}_i - \bar{z}_j)^{2k - \frac{1}{m}} \tag{4.3.2}$$

where k is again a positive integer. The exchange properties are the same as for quasiholes. Numerical studies [11] have shown that (4.3.2) is in good correspondence with the exact quasiparticle ground state.

The plasma analogy $|\psi|^2 = \exp(-V)$ is used to find the electron filling fraction ν' of the states (4.3.1) and (4.3.2). Using $|\bar{z}_i - \bar{z}_j| = |z_i - z_j|$ we get

$$V = \frac{4\pi}{m(2km + \alpha)} \left[\frac{2km + \alpha}{4\pi} \sum_k |z_k|^2 - \frac{(2km + \alpha)^2}{2\pi} \sum_{i<j} \ln |z_i - z_j| \right] \tag{4.3.3}$$

where $\alpha = +1$ for quasiholes and $\alpha = -1$ for quasiparticles. The expression between square brackets is the potential energy of a plasma of particles with 'charge' $2km + \alpha$ in a neutralising background of uniform density $1/\pi\ell^2$. The condition of neutrality gives the following excitation density

$$n_{exc} = \frac{1}{(2km + \alpha)\pi\ell^2}. \tag{4.3.4}$$

As the excitations have electric charge $q_{exc} = \alpha\frac{e}{m}$, we see that the presence of excitations changes the electron density by an amount

$$\triangle n_e = -(q_{exc}/e)n_{exc}. \tag{4.3.5}$$

The new electron filling fraction ν' is

$$\nu' = \nu + \frac{h}{eB}\triangle n_e = \frac{1}{m} - \frac{\alpha}{m(2km + \alpha)} = \frac{1}{m + \frac{\alpha}{2k}}. \tag{4.3.6}$$

When this process of creating quantum fluids on top of quantum fluids is iterated, the result is a continued fraction

$$\nu = \frac{1}{m + \frac{\alpha_1}{2k_1 + \frac{\alpha_2}{2k_2 + \cdots}}} \tag{4.3.7}$$

where $k_i \in N$ and α_i is $+1(-1)$ when quasiholes(particles) are created. The values for ν obtained in this way always have odd denominators.

For every observed fraction all parent states are also observed. Note that the IQHE is obtained by the creation of extra particle excitations from the $\nu = 1$ ground state.

Although all observed fractions are reproduced in this scheme, it is troublesome that we need eight iterations and that with less than eight iterations many fractions are obtained where the FQHE has not been observed. It seems unlikely that the superposition of many quasiparticle ground states on top of each other at high quasiparticle densities is a correct picture of the true physics of a quantum Hall system.

4.3.2 Jain's composite fermion hierarchy

Jain [14] proposed that the fractional effect is a manifestation of the integer effect of electrons or holes that are bound to an even number of flux quanta. Such charge/flux composites have fermionic statistics and are therefore called *composite fermions*. It is assumed that the composite fermion Coulomb interaction is very weak compared to the electron-electron repulsion. In this approach the stability of the fractional effect is inherited from the stability of completely filled Landau levels.

The integer QHE of the composite fermions occurs when the filling fraction *of the composite fermions* has an integer value. The *electron* filling $\nu := \frac{n_e\Phi_0}{B}$ is fractional at those points, since in a mean field sense the flux tubes correspond to an extra contribution to the magnetic field B. All the experimentally found fractions can be obtained by choosing appropriate values for the integer filling fraction of composite fermions and the amount of flux that these composite fermions carry.

When the IQHE at filling fraction m occurs for electrons that carry $2p$ flux quanta ($p \in N$), there are two contributions to the magnetic field: The field $\Phi_0 n_e/m$ needed to get filling m, and the mean field contribution $2p\Phi_0 n_e$ due to the fluxes. These do not necessarily have the same sign,

$$B = (2p \pm \frac{1}{m})\Phi_0 n_e. \tag{4.3.8}$$

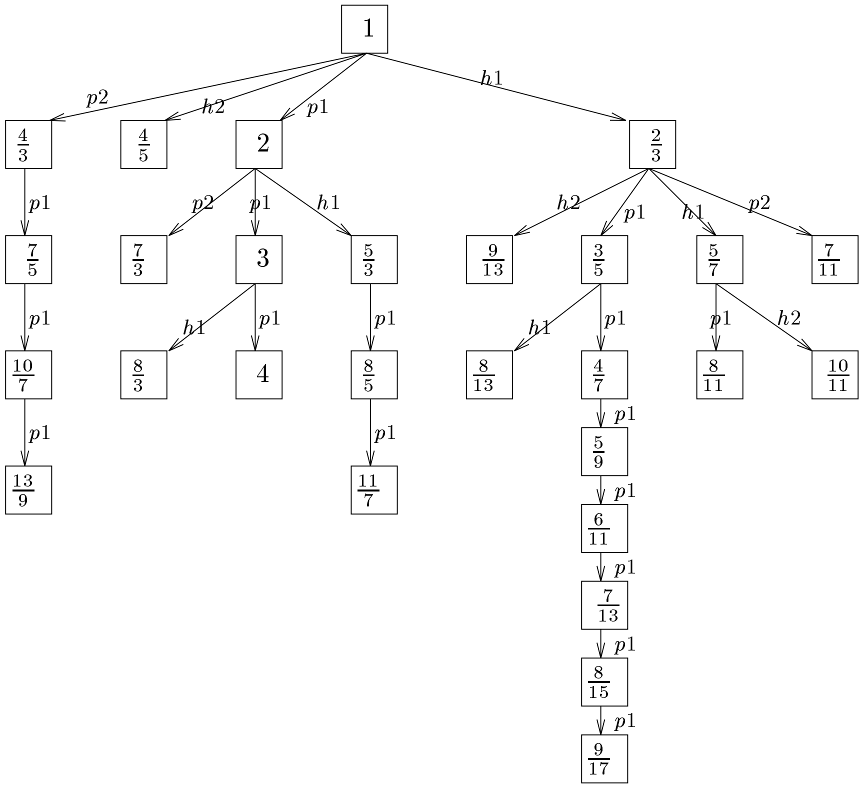

Figure 4.3.1: A part of the Haldane-Halperin hierarchy, starting from $\nu = 1$. Only observed fractions are shown. Quasiparticle(hole) excitations are denoted by p(h). The integer after this letter is the value of k.

The electron filling fraction of this system is (see figure 4.3.2)

$$\nu(m,p) = \frac{n_e \Phi_0}{B} = \frac{m}{2pm \pm 1}. \tag{4.3.9}$$

All these fractions have odd denominators and lie in the lowest Landau level. Due to electron-hole symmetry, the FQHE also manifests itself at the following fillings

$$\tilde{\nu}(m,p) = 1 - \nu(m,p). \tag{4.3.10}$$

Fractions like $\frac{4}{3}$ and $\frac{8}{3}$, not lying in the LLL, are obtained by starting with a system at filling $\nu(m,p)$ or $\tilde{\nu}(m,p)$ and adding an integer number of filled Landau levels to it.

The values found in this way are the first level of a hierarchy: We can repeat the step of attaching flux tubes to the particles that form a stable state. This modification alters the filling fraction just as in (4.3.9),

$$\nu \to \frac{\nu}{2p\nu \pm 1}. \tag{4.3.11}$$

All resulting fractions have odd denominators.

It is natural to assume that any deviation from the mean field picture will cause less stable plateaus. Therefore fractions obtained after several iterations will be less stable than their parent states, and the addition $\nu \to \nu+\text{int}$ will drastically reduce the stability. The stability also decreases with increasing amounts of bound flux, since it is hard for electrons to monopolize an area big enough to bind much flux.

The composite fermion model has several advantages over the Haldane-Halperin hierarchy.

- Most of the experimentally observed fractions are obtained in the *first* level of the composite fermion hierarchy.
- The stabilities of the FQHE plateaux are predicted very well for the first level of the hierarchy. The composite fermion model also predicts which fractions will be observed next when sample quality is improved.
- There is experimental evidence for the existence of composite fermions at the series of fractions $\frac{m}{2m\pm1}$.

In the following chapters we will discuss only Jain's hierarchy.

It is not known what mechanism could account for the binding of magnetic flux to electrons. The electron-electron repulsion is thought to play an essential role, since it allows electrons to keep each other away from a certain area and the flux through it.

Figure 4.3.2: The first level of Jain's hierarchy. The fractions $\frac{m}{2pm-1}$ and $\frac{m}{2pm+1}$ are shown on the left and the right half, respectively. The filled circles show the fractions that have been observed. The empty circles show the predicted next most stable fractions.

Chapter 5

From Chern-Simons theory to edge bosons

Rekenen in het openbaar is obsceen.

G.G.A. Bäuerle

In this chapter, we will not try to find out why σ_H takes certain values, but we will consider it as given and then describe the quantum Hall effect at that plateau.

We will first show that equation (2.2.9), giving the microscopic flow of electric current in an ideal sample, is derived from the action of a Chern-Simons gauge field coupled to the electric current. It is well known that such an action describes anyonic particles. In this way the quasiparticles/holes of chapter 4 are found again.

Then, following Fröhlich and Kerler [20], we show that the phenomenology at the Hall plateaus can be described by a pure Chern-Simons action, and that this can be rewritten as an action on the edge of the sample. The edge currents are found to be chiral and to satisfy a $U(1)$ Kač-Moody algebra.

At that point we recall some of the basic notions of conformal field theory in $1+1$ dimensions and show the equivalence of fermions and bosons. Using the representation theory of the Virasoro algebra, the spectrum of edge particles is determined. Composite fermions and fractionally charged quasiparticles are obtained in this way. The Laughlin trial wave functions are recovered as the N-point correlation functions of fermionic vertex operators.

We conclude by showing how Jain's hierarchy scheme can be implemented by combining several edge boson fields. Trial wave functions for the states in this hierarchy are constructed.

5.1 Chern-Simons theory

Equation (2.2.9) describing the microscopic flow of current in an ideal quantum Hall sample can be written as

$$j^i = \sigma_H \varepsilon^{ik} E_k \tag{5.1.1}$$

where $i = 1, 2$ and ε^{ik} is the Levi-Civita tensor in two dimensions. Using Maxwell's equations, the current conservation $\nabla \cdot \vec{\jmath} + \partial_t \rho = 0$ and (5.1.1), one finds

$$\sigma_H \partial_t B = -\sigma_H \varepsilon^{ik} \partial_i E_k = -\partial_i j^i = \partial_t \rho. \tag{5.1.2}$$

This relation between ρ and B can be written as

$$\exists \rho_{bound}(\vec{x}) \qquad j^0(x^\mu) := \rho(x^\mu) - \rho_{bound}(\vec{x}) = \sigma_H B(x^\mu) \tag{5.1.3}$$

The *mobile* charge density is coupled to the magnetic field. If the magnetic field is constant in time, the electron gas is incompressible. Here we see that the aspect of incompressibility is present in the very equations of motion of the QHE.

Equations (5.1.1) and (5.1.3) can be written together as a (2+1) dimensional relativistically covariant equation of motion,

$$j^\mu = -\frac{\sigma_H}{2}\varepsilon^{\mu\nu\sigma}F_{\nu\sigma} \tag{5.1.4}$$

or, in the language of differential forms,

$$j = -\sigma_H * dA. \tag{5.1.5}$$

where * is the Hodge star. This equation of motion is obtained from the action

$$S = \int d^3x[-j_\mu A^\mu - \frac{\sigma_H}{2}\varepsilon^{\mu\nu\kappa}A_\mu\partial_\nu A_\kappa] \tag{5.1.6}$$

by varying A^μ. The second term on the r.h.s. is called a *Chern-Simons* action [15]. It is well known that a Chern-Simons gauge field coupled to a current describes anyons. The appearance of a Chern-Simons term in the action is not surprising, since in chapter 4 anyons were found to play a role in the FQHE. Equation (5.1.3), which is derived solely from the current flow (5.1.1), shows that charge/flux composites already appear at a very basic level of description in the QHE. However, in this phenomenological picture *all* flux is pinned on the mobile charges.

If the current j^μ is due to point particles with charge $-e$, equation (5.1.3) tells us that there are ν^{-1} flux quanta at the location of every electron. According to (4.2.21) such charge/flux composites have the following statistics

$$s_\nu = \frac{1}{2\nu}. \tag{5.1.7}$$

It is interesting to note that the electromagnetic field aquires a mass σ_H in this description of the QHE (see [15]). If we combine (5.1.4) with the Maxwell equations $\partial_\kappa F^{\kappa\mu} = j^\mu$, we get

$$\partial_\kappa F^{\kappa\mu} = -\frac{\sigma_H}{2}\varepsilon^{\mu\nu\sigma}F_{\nu\sigma} \tag{5.1.8}$$

yielding a massive wave equation

$$(\partial_\kappa\partial^\kappa + \sigma_H^2)F_{\mu\nu} = 0. \tag{5.1.9}$$

An important consequence is that even in a quantized theory A_μ can be treated as a classical field at large distance scales.

5.1.1 Non-relativistic Chern-Simons field theory

Let us view the fields appearing in the action (5.1.6) as quantum operators in a non-relativistic theory. In terms of the scalar electron field operator ψ, the electric current j^μ is given by

$$\begin{aligned} j^0 &= -e\psi^\dagger\psi \\ \vec{j} &= -e\frac{\hbar}{2mi}[\psi^\dagger\vec{D}\psi - h.c.] \end{aligned} \tag{5.1.10}$$

and the action reads

$$S[\psi, A^\mu] = \int d^3x \left[i\hbar\psi^\dagger D_0\psi + \frac{\hbar^2}{2m}(D_i\psi)^\dagger(D^i\psi) - \frac{\sigma_H}{2}\varepsilon^{\mu\nu\kappa}A_\mu\partial_\nu A_\kappa \right]. \tag{5.1.11}$$

We will show that just as in the case of classical point-particles, A^μ can be formally written as a pure gauge. We then perform a gauge transformation that eliminates A^μ and show that the transformed electron field operator $\tilde{\psi}$ no longer satisfies anticommutation relations but an anyonic braiding relation. This is the analogue of the altered boundary conditions (4.2.24).

We write (5.1.4) in the form

$$\nabla \times \vec{A} = \frac{1}{\sigma_H} j^0 \tag{5.1.12}$$

$$\partial_0 A_i - \partial_i A_0 = -\frac{1}{\sigma_H}\varepsilon_{ik} j^k. \tag{5.1.13}$$

Equation (5.1.12) is easily solved,

$$A^i(\vec{x}) = -\frac{1}{\sigma_H}\int d^2y j^0(\vec{y})\epsilon^{ij}\frac{\partial}{\partial x_j}G(\vec{x}-\vec{y}) = \frac{1}{2\pi\sigma_H}\int d^2y j^0(\vec{y})\frac{\partial}{\partial x^i}\Phi(\vec{x}-\vec{y}). \tag{5.1.14}$$

Special care must be taken in moving the derivative $\partial/\partial x^i$ out of the second integral, since the angle function $\Phi(\vec{x}-\vec{y})$ is multi-valued. In order to make Φ single-valued, a branch-cut has to be specified in the y-plane starting at $\vec{x}$. Wherever the cut is placed, the range of y-integration will depend on $\vec{x}$ so that $\partial/\partial x^i$ cannot be moved out of the integral. However, Jackiw and Pi [12] showed that it can be done *if* j^0 is a sum of δ-functions. In our non-relativistic theory this is the case. We can formally write $\vec{A}$ as a total derivative

$$\vec{A}(\vec{x}) = \nabla\left(\frac{1}{2\pi\sigma_H}\int d^2y\Phi(\vec{x}-\vec{y})j^0(\vec{y})\right). \tag{5.1.15}$$

Now we show that A^0 can also be written as a total derivative. Acting on (5.1.13) with ∂^i and using $\nabla\cdot\vec{A} = 0$ and the continuity equation we get

$$A_0(\vec{x}) = \frac{1}{\sigma_H}\int d^2y G(\vec{x}-\vec{y})\nabla\times\vec{j}(\vec{y}) = \frac{1}{2\pi\sigma_H}\int d^2y j^i(\vec{y})\frac{\partial}{\partial x^i}\Phi(\vec{x}-\vec{y}). \tag{5.1.16}$$

Integrating by parts and using the continuity equation again, we find

$$A_0(\vec{x}) = -\frac{1}{2\pi\sigma_H}\int d^2y\Phi(\vec{x}-\vec{y})\partial_0 j^0(\vec{y}) = \partial_0\left(-\frac{1}{2\pi\sigma_H}\int d^2y\Phi(\vec{x}-\vec{y})j^0(\vec{y})\right) \tag{5.1.17}$$

Now A_μ can be formally written as a total derivative

$$A_\mu = s\partial_\mu\Lambda \quad ; \quad s = \frac{1}{2\nu} \tag{5.1.18}$$

with

$$\Lambda(\vec{x}) = -2\frac{\hbar}{e^2}\int d^2y\Phi(\vec{x}-\vec{y})j^0(\vec{y}) = 2\frac{\hbar}{e}\int d^2y\Phi(\vec{x}-\vec{y})\psi^\dagger(\vec{y})\psi(\vec{y}). \tag{5.1.19}$$

Just as was the case in chapter 4, the topological interaction via the gauge field A^μ can be traded for complicated boundary conditions, but now not on wave functions but on field operators. The gauge field A^μ is eliminated by a singular gauge transformation

$$\begin{aligned} A_\mu &\rightarrow A_\mu - s\partial_\mu \Lambda = 0 \\ \psi(\vec{x}) &\rightarrow \tilde{\psi}(\vec{x}) := e^{-i\frac{e}{\hbar}s\Lambda(\vec{x})}\psi(\vec{x}) = e^{i2s\int d^2y\ \Phi(\vec{x}-\vec{y})j^0(\vec{y})}\psi(\vec{x}). \end{aligned} \tag{5.1.20}$$

The action takes the form of a free action

$$S = \int d^3x \left[i\hbar\tilde{\psi}^\dagger\partial_0\tilde{\psi} + \frac{\hbar^2}{2m}(\partial_i\tilde{\psi})^\dagger(\partial^i\tilde{\psi}) \right]. \tag{5.1.21}$$

The effects of the Chern-Simons action are now completely encoded in the new anyon field $\tilde{\psi}$. Using the canonical equal-time anticommutation relations

$$\begin{aligned} \{\psi(\vec{x}), \psi^\dagger(\vec{y})\} &= \delta(\vec{x}-\vec{y}) \\ \{\psi(\vec{x}), \psi(\vec{y})\} &= \{\psi^\dagger(\vec{x}), \psi^\dagger(\vec{y})\} = 0 \end{aligned} \tag{5.1.22}$$

and the fact that

$$[A, BC] = \{A, B\}C - B\{C, A\} \tag{5.1.23}$$

we get the following commutators of Λ

$$\begin{aligned} [\Lambda(\vec{x}), \psi(\vec{y})] &= -2\frac{\hbar}{e}\Phi(\vec{y}-\vec{x})\psi(\vec{y}) \\ [\Lambda(\vec{x}), \Lambda(\vec{y})] &= 0. \end{aligned} \tag{5.1.24}$$

Now we can determine the anyonic 'commutation' relations of the field $\tilde{\psi}$. For $\vec{x} \neq \vec{y}$ we get

$$\begin{aligned} \tilde{\psi}(\vec{x})\tilde{\psi}(\vec{y}) &+ e^{i2s[\Phi(\vec{x}-\vec{y})-\Phi(\vec{y}-\vec{x})]}\tilde{\psi}(\vec{y})\tilde{\psi}(\vec{x}) = 0 \\ \tilde{\psi}(\vec{x})\tilde{\psi}^\dagger(\vec{y}) &+ e^{-i2s[\Phi(\vec{x}-\vec{y})-\Phi(\vec{y}-\vec{x})]}\tilde{\psi}^\dagger(\vec{y})\tilde{\psi}(\vec{x}) = \delta(\vec{x}-\vec{y}). \end{aligned} \tag{5.1.25}$$

These are braiding relations. For $\vec{x} = \vec{y}$ the standard anticommutation relations remain unchanged, because then $\Phi(\vec{x}-\vec{y}) - \Phi(\vec{y}-\vec{x})$ vanishes.

With the choice we have made for the angle function Φ, namely $\Phi \in (-\pi, \pi]$, an anticlockwise exchange of $\vec{x}$ around $\vec{y}$ gives

$$\Phi(\vec{x}-\vec{y}) - \Phi(\vec{y}-\vec{x}) = -\pi, \tag{5.1.26}$$

The braiding relations (5.1.25) for $\vec{x} \neq \vec{y}$ become

$$\begin{aligned} \tilde{\psi}(\vec{x})\tilde{\psi}(\vec{y}) &+ q\tilde{\psi}(\vec{y})\tilde{\psi}(\vec{x}) = 0 \\ \tilde{\psi}(\vec{x})\tilde{\psi}^\dagger(\vec{y}) &+ q^*\tilde{\psi}^\dagger(\vec{y})\tilde{\psi}(\vec{x}) = \delta(\vec{x}-\vec{y}) \end{aligned} \tag{5.1.27}$$

with $q := e^{-i2\pi s}$.

5.1.2 The pure Chern-Simons action

In 1990 Fröhlich and Kerler [20] showed that the QHE at the plateaux can in fact be described by a *pure* Chern-Simons action. This has far-reaching consequences: The theory of the QHE effectively becomes an edge theory of incompressible quantum liquids.

If we define a two-form J as the Hodge dual of j,

$$J := *j, \tag{5.1.28}$$

then (5.1.5) can be written as

$$J = -\sigma_H dA. \tag{5.1.29}$$

From this it is clear that we can define a 'vector potential' β for the electric current, such that

$$J =: d\beta. \tag{5.1.30}$$

We can now write (5.1.5) as

$$d\beta = -\sigma_H dA. \tag{5.1.31}$$

This equation of motion is obtained from the pure Chern-Simons action

$$S_{CS} = -\frac{1}{2\sigma_H}\int(\beta+\sigma_H A)\wedge d(\beta+\sigma_H A). \tag{5.1.32}$$

The factor $-(2\sigma_H)^{-1}$ has been put in front in order to make contact with (5.1.6), where the interaction term is $-\int d^3x j^\mu A_\mu = -\int A\wedge d\beta$.

The Chern-Simons action has been the subject of intense study for many years, for it has lots of interesting properties. First of all, the pure C-S action is topological: it does not depend on a choice of a metric or coordinates. Thus it pops up in the action of a physical theory whenever topological effects are present, e.g. 2+1 dimensional gravity, topological field theory, anyons.

Another property of the pure C-S action on a manifold M is that edge currents arise on ∂M. This was first discovered by Witten [21].

5.2 Edge currents; U(1) Kač-Moody algebra

We will now explain the relation between the Chern-Simons action on a 3D manifold $\Sigma\times R$ and edge currents on $\partial\Sigma\times R$. We do this in two ways: First we will use the canonical formalism, following the approach of Balachandran [16]. Then we will use path integrals, following Fröhlich and Kerler [20].

Let us first have a look at the symmetries of the Chern-Simons action. It is convenient to introduce the one-form a, defined as

$$a := \frac{e}{\hbar}(\sigma_H^{-1}\beta + A), \tag{5.2.1}$$

in order to get the most commonly used (dimensionless) form of the Chern-Simons action,

$$S_{CS}[a] = -\frac{\nu}{4\pi}\int_{\Sigma\times R} a\wedge da = -\frac{\nu}{4\pi}\int_{\Sigma\times R} d^3x\varepsilon^{\mu\kappa\lambda}a_\mu\partial_\kappa a_\lambda. \tag{5.2.2}$$

We have written 'ν' in front of the integral, but this should be read as $\frac{h}{e^2}\sigma_H$. What is meant is the filling fraction *at the center* of a Hall plateau.

The action (5.2.2) is invariant under the gauge transformations

$$a \to a + d\chi \;, \qquad \chi|_{\partial\Sigma} = 0. \tag{5.2.3}$$

The conserved Noether current is

$$J^{\mu}_{Noether} = \frac{\nu}{4\pi}\varepsilon^{\mu\kappa\lambda}[2\chi\partial_{\kappa}a_{\lambda} - \partial_{\kappa}(\chi a_{\lambda})] \tag{5.2.4}$$

and the conserved charge that generates the gauge transformation (5.2.3) is

$$Q[\chi] = \int_{\Sigma} d^2x J^0_{Noether} = \frac{\nu}{2\pi}\int_{\Sigma}\chi da. \tag{5.2.5}$$

5.2.1 The canonical formalism; Constraint analysis

If one wants to canonically quantize a classical theory, one has to set up a Poisson bracket structure on the phase space and then replace the Poisson brackets by (anti)commutators, the phase space by a Hilbert space of quantum states, and functions on phase space by operators on Hilbert space. This program can be complicated by the appearance of constraints on phase space.

Let us denote the spatial coordinates in Σ by x^1, x^2 and the time by x^0. The canonical momenta conjugate to a^{μ} are

$$\begin{aligned}
\pi_i \;&:=\; \frac{\partial L}{\partial(\partial_0 a^i)} = -\frac{\nu}{4\pi}\varepsilon_{ij}a^j, \qquad i,j = 1,2 \\
\pi_0 \;&:=\; \frac{\partial L}{\partial(\partial_0 a^0)} = 0.
\end{aligned} \tag{5.2.6}$$

The first equation tells us that the components a_1 and a_2 are canonically conjugate, so that the equal-time Poisson bracket becomes

$$\begin{aligned}
\{A,B\} \;&=\; -\frac{2\pi}{\nu}\varepsilon^{ij}\int_{\Sigma} d^2x \frac{\delta A}{\delta a^i(\vec{x})}\frac{\delta B}{\delta a^j(\vec{x})} \\
&+\; \int_{\Sigma} d^2x \left[\frac{\delta A}{\delta a_0(\vec{x})}\frac{\delta B}{\delta \pi_0(\vec{x})} - \frac{\delta B}{\delta a_0(\vec{x})}\frac{\delta A}{\delta \pi_0(\vec{x})}\right]
\end{aligned} \tag{5.2.7}$$

Equation (5.2.6) shows us that $\partial_0 a_0$ does not appear in the action. There is no dynamical equation for a_0, so a_0 is not a dynamical variable of the system.

The equation $\pi_0 = 0$ is a constraint on the phase space of the system. The actual phase space is spanned by the set $\{a^i, a^0, \pi_0\}$, but the physical part $\mathcal{C}$ of it satisfies $\pi_0 = 0$. We cannot simply put π_0 identically equal to zero, since then we would never be able to obtain the canonical equal-time Poisson bracket

$$\{a^0(\vec{x}), \pi_0(\vec{y})\} = \delta(\vec{x} - \vec{y}). \tag{5.2.8}$$

If $\mathcal{C}$ is defined as that part of the phase space where a set of constraint functions $\{C_i\}$ is zero, then it is said that the C_i are *weakly* zero. The notation for this weak equality is $C_i \approx 0$.

A system is subjected to constraints by adding the constraint functions to the Hamilton density,

$$\begin{aligned}
\mathcal{H} \;&=\; \dot{q}_i\frac{\partial L}{\partial \dot{q}_i}(q,\dot{q}) - L(q,\dot{q}) + \lambda^j C_j(q,\pi) \\
&=:\; \mathcal{H}_0(q,\pi) + \lambda^j C_j(q,\pi),
\end{aligned} \tag{5.2.9}$$

where the λ^j are as yet undetermined Lagrange multipliers, and H_0 can be simplified by using $C_j \approx 0$.

Constraints are preserved in time. Therefore we have

$$\{C_j, H\} \approx 0. \tag{5.2.10}$$

This may determine some of the Lagrange multipliers, but it can also lead to new constraints

$$C'_j(q, \pi) \approx 0. \tag{5.2.11}$$

These new constraints also have to satisfy $\{C'_j, H\} \approx 0$, determining more of the Lagrange multipliers and/or leading to even more constraints. This procedure is continued until no more new constraints are generated.

The resulting collection of constraints can be divided into first class and second class constraints. First class constraints F_i are those that satisfy

$$\{F_i, C_j\} \approx 0, \qquad \forall j \tag{5.2.12}$$

In other words, canonical transformations generated by first class constraints preserve all constraints, i.e. they map $\mathcal{C}$ into $\mathcal{C}$. The remaining constraints S_i are defined to be second class. They generate canonical transformations that lift points in $\mathcal{C}$ out of $\mathcal{C}$.

An important property of first class constraints is that they generate gauge transformations: Imagine a path in $\mathcal{C}$ representing the time evolution of the physical system given by H. Since the F_j satisfy (5.2.10) one is free to perform a canonical transformation generated by the F_j at any moment in time without changing the Hamiltonian c.q. the equations of motion. This can be done in many ways. The resulting paths in $\mathcal{C}$ are all equivalent: We have obtained an equivalence class of paths that can be mapped into each other by gauge transformations. It can be shown that the first class constraints form a closed algebra,

$$\{F_i, F_j\} = c_{ij}^k F_k, \tag{5.2.13}$$

so that the gauge transformations form a group.

Classical 'observables' $\mathcal{O}$ can now be defined as those functions on $\mathcal{C}$ that are gauge invariant,

$$\delta_{F_j}\mathcal{O} := \{\mathcal{O}, F_j\} \approx 0. \tag{5.2.14}$$

On the right-hand side we could have a combination of F's and G's. However, observables should also satisfy $\delta_{F_i}(\delta_{F_j}\mathcal{O}) \approx 0$, which amounts to requiring that (5.2.14) has the more specific form

$$\{\mathcal{O}, F_j\} = d_j^i F_i. \tag{5.2.15}$$

Let us see what constraints we have in the case of the Chern-Simons action. We already saw that $\pi_0 \approx 0$. The other constraints are found by imposing time independence $\dot{\pi}_0 \approx 0$. The Hamiltonian following from (5.2.2) is given by

$$H_0 = \frac{\nu}{2\pi} \int_\Sigma a_0 da \tag{5.2.16}$$

This gives

$$0 \approx \{\pi_0(\vec{x}), H\} = -\frac{\nu}{2\pi} f_{12}(\vec{x}). \tag{5.2.17}$$

We have found a new constraint, namely $f_{12} \approx 0$. Requiring time independence of f_{12} does not lead to more constraints, as $\{f_{12}, H\} = 0$. The result is that we have two first class constraints, (since $\{\pi_0, f_{12}\} = 0$), and no second class ones.

The gauge transformations generated by these constraints are not independent, because of relation (5.2.17). Let us define

$$g[\chi] := \frac{\nu}{2\pi} \int_\Sigma d^2x \chi(\vec{x}) f_{12}(\vec{x}) \tag{5.2.18}$$

with χ a well-behaved function (we will see what that means in a moment). The gauge transformations generated by $g[\chi]$ are

$$\begin{aligned} \delta_g a_0(\vec{x}) &= \{a_0(\vec{x}), g_0[\chi]\} \\ &= \int_\Sigma d^2y \chi(\vec{y}) \{a_0(\vec{x}), \{\pi_0(\vec{y}), H\}\} = \partial_0 \chi(\vec{x}) \\ \delta_g a_i(\vec{x}) &= \{a_i(\vec{x}), g[\chi]\} = \partial_i \chi(\vec{x}). \end{aligned} \tag{5.2.19}$$

The gauge transformations have to form a closed algebra. This requirement imposes an important restriction on the functions χ; Using (5.2.7) and (5.2.18) we find

$$\{g[\chi_1], g[\chi_2]\} = \frac{\nu}{2\pi} \int_{\partial\Sigma} \chi_1 d\chi_2. \tag{5.2.20}$$

In order to ensure that the g's form a closed algebra, we have to impose $\chi|_{\partial\Sigma} = 0$, so that $\{g[\chi_1], g[\chi_2]\} = 0$. In this way we have reobtained the gauge transformations (5.2.3).

The observables of the theory are by definition those functions on $\mathcal{C}$ that satisfy (5.2.15). They are the gauge invariant field strengths $f_{\mu\kappa}$, integrated with an arbitrary function on Σ.

$$\begin{aligned} \mathcal{O}_{\mu\kappa}[\Lambda] &:= \frac{\nu}{2\pi} \int_\Sigma d^2x \Lambda(x) f_{\mu\kappa}(x) \\ \{\mathcal{O}_{\mu\kappa}[\Lambda], g[\chi]\} &= 0. \end{aligned} \tag{5.2.21}$$

An important fact is that two observables $\mathcal{O}_{12}[\Lambda_1]$ and $\mathcal{O}_{12}[\Lambda_2]$ are equivalent when Λ_1 and Λ_2 are identical on the edge, since in that case we have

$$\mathcal{O}_{12}[\Lambda_1] - \mathcal{O}_{12}[\Lambda_2] = g[\Lambda_1 - \Lambda_2] \approx 0. \tag{5.2.22}$$

We actually have to define equivalence classes $\overline{\mathcal{O}_{12}[\Lambda]}$ that depend only on $\Lambda|_{\partial\Sigma}$:

$$\overline{\mathcal{O}_{12}[\Lambda]} := \{\mathcal{O}_{12}[\Lambda + \chi] \ , \quad \chi|_{\partial\Sigma} = 0\}. \tag{5.2.23}$$

This is a very important result. It shows that the complete physics of the QHE at the plateaus is contained in the dynamics on the edge.

The poisson brackets of the observables with themselves are

$$\begin{aligned} \{\mathcal{O}_{0i}[\Lambda_1], \mathcal{O}_{\mu\kappa}[\Lambda_2]\} &= 0 \\ \{\mathcal{O}_{12}[\Lambda_1], \mathcal{O}_{12}[\Lambda_2]\} &= \frac{\nu}{2\pi} \int_{\partial\Sigma} \Lambda_1 d\Lambda_2. \end{aligned} \tag{5.2.24}$$

We recognize the second relation as a $U(1)$ Kač-Moody algebra. In the next sections we will see that this is the current algebra of a free massless chiral boson or fermion field on $\partial\Sigma$, and that the chirality is determined by the sign of ν.

5.2.2 Path integral approach; Gauge fixing

The equivalence of the observables in pure CS-theory and a massless chiral boson field on $\partial\Sigma \times R$ can be nicely shown by the use of path integrals.

Because of the gauge invariance of $S_{CS}[a]$ one cannot just take functional integrals over a. Instead, one should integrate over a modulo gauge transformations. One way of doing this is is to choose a gauge leaving no gauge freedom, and then perform all path integrations in that gauge. This is called *gauge fixing*. The results are independent of the chosen gauge.

In general, if we have a $2n$-dimensional phase space and there are m first class constraints F_i then the trajectories generated by the F_i span an m-dimensional equivalence class $\mathcal{E}$ in the $2n-m$ dimensional constraint surface $\mathcal{C}$. m gauge fixing constraints g_i are needed to eliminate all degrees of gauge freedom. The submanifold $\mathcal{C}/\mathcal{E}$ is defined by the constraints $F_i \approx 0$ and $g_i \approx 0$. It is the intersection of the two constraint surfaces. The g's should be chosen is such a way that every equivalence class $\mathcal{E}$ is represented by a single point in $\mathcal{C}/\mathcal{E}$. It can be shown that for this property to hold, a sufficient condition is that

$$\det\{g_i, f_j\} \neq 0 \quad , \quad i,j = 1, \cdots, m. \tag{5.2.25}$$

A canonical transformation $(p,q) \to (P,Q)$ can be performed such that

$$(P_{n-m+1}, \cdots, P_n) = (g_1, \cdots, g_m) \tag{5.2.26}$$

Since there are only $2n-m$ independent coordinates on $\mathcal{C}$, m coordinates have to depend on these independent ones. We can make the choioce

$$Q_j|_{\mathcal{C}} = Q_j(Q_1, \cdots, Q_{n-m}, P_1, \cdots, P_n) \quad , \quad j = n-m+1, \cdots, n. \tag{5.2.27}$$

This dependence can be viewed as a definition of $\mathcal{C}$. We obtain $\mathcal{C}/\mathcal{E}$ by setting

$$P_{n-m+1}, \cdots, P_n = 0. \tag{5.2.28}$$

Path integrals are very simple now, since only the independent variables have to be integrated over,

$$< f|i >= \int_i^f \mathcal{D}(P,Q) e^{\frac{i}{\hbar} S[P,Q]}. \tag{5.2.29}$$

However, it is in general impractical to perform the canonical transformation $(p,q) \to (P,Q)$. It is more practical to express (5.2.29) in terms of (p,q). The integration measure can be written as

$$\prod_{k=1}^{n-m} dQ_k dP_k = [\prod_{k=1}^{n} dQ_k dP_k] \prod_{j=n-m+1}^{n} \delta(P_n)\delta[Q_j - Q_j(Q_1, \cdots, Q_{n-m}, P_1, \cdots, P_n)]. \tag{5.2.30}$$

Since (5.2.27) defines $\mathcal{C}$, the second product of δ-functions in (5.2.30) is the same (apart from a Jacobian) as the set of first class constraints F_i.

$$\prod_{j=n-m+1}^{n} \delta[Q_j - Q_j(Q_1, \cdots, Q_{n-m}, P_1, \cdots, P_n)] = \frac{\partial(F_1, \cdots, F_m)}{\partial(Q_{n-m+1}, \cdots, Q_n)} \prod_{k=1}^{m} \delta[F_k(p,q)]. \tag{5.2.31}$$

From (5.2.26) it follows that the Jacobian is in fact $\det\{F_i, g_j\}_{QP} = \det\{F_i, g_j\}_{qp}$. Using $P_{n-m+j} = g_j(p, q)$ we can finally rewrite (5.2.29) as

$$< f|i >= \int_i^f \mathcal{D}(p, q) \det\{F, g\}[\prod_{i=1}^m \delta(g_i)\delta(F_i)]e^{\frac{i}{\hbar}S[p,q]}. \tag{5.2.32}$$

Now let us see what this gauge fixing procedure yields in the case of our Chern-Simons action. If we use a radial coordinate u and light-cone coordinates $u^{\pm} := vt \pm \theta$, $\partial_{\pm} := \frac{1}{v}\partial_t \pm \partial_\theta$ with θ a dimensionful angle coordinate and v an arbitrary positive constant with the dimension of a speed, we can write

$$\begin{aligned} d = d_- + \underline{d} &:= (du^-\partial_-) + (du^+\partial_+ + du\partial_u) \\ a = a_- + \underline{a} &:= (a_{u_-}du^-) + (a_{u_+}du^+ + a_u du). \end{aligned} \tag{5.2.33}$$

The action (5.2.2) takes the form

$$S_{CS}[a] = -\frac{\nu}{4\pi}\int_{\Sigma\times R}(a_- \wedge \underline{d}\,\underline{a} + \underline{a} \wedge da). \tag{5.2.34}$$

In the first term a_- appears as a multiplier enforcing the constraint $\underline{d}\,\underline{a} = 0$. A nice way of gauge fixing is to choose the light cone gauge $a_- = 0$. This choice still leaves some residual gauge freedom,

$$a \to a + \underline{d}\chi, \qquad \chi|_{\partial\Sigma} = 0, \tag{5.2.35}$$

but we will see in a moment that this freedom is irrelevant. In the light cone gauge, the path integral (5.2.32) takes the form

$$\begin{aligned} < f|i > &= \int_i^f \mathcal{D}[a]\delta(a_-)\delta(\underline{d}\,\underline{a})\{\underline{d}\,\underline{a}, a_-\}e^{\frac{i}{\hbar}S_{CS}[a]} \\ &= \int_i^f \mathcal{D}[a]\delta(a_-)\delta(\underline{d}\,\underline{a})\{\underline{d}\,\underline{a}, a_-\}\exp\left(-\frac{i}{\hbar}\frac{\nu}{4\pi}\int_{\Sigma\times R}\underline{a} \wedge d_-\underline{a}\right). \end{aligned} \tag{5.2.36}$$

The Poisson bracket yields a constant in phase space. The constraints $a_- = 0$ and $\underline{d}\,\underline{a} = 0$ can be implemented as follows: We define a $U(1)$-field g on $\Sigma \times R$,

$$g = e^{-i\varphi} \quad ; \quad a = ig^{-1}dg = d\varphi, \tag{5.2.37}$$

where φ is a scalar field satisfying $\partial_-\varphi = 0$. The constraints are now written as

$$\delta(a_-)\delta(\underline{d}\,\underline{a}) = \delta(a - d\varphi)\delta(\partial_-\varphi), \tag{5.2.38}$$

and the path integral becomes an integral over φ,

$$\begin{aligned} < f|i > &\propto \int_i^f \mathcal{D}[\varphi]\delta(\partial_-\varphi)\exp\left(-\frac{i}{\hbar}\frac{\nu}{4\pi}\int_{\Sigma\times R}\underline{d\varphi} \wedge d_-\underline{d\varphi}\right) \\ &= \int_i^f \mathcal{D}[\varphi]\delta(\partial_-\varphi)\exp\left(-\frac{i}{\hbar}\frac{\nu}{4\pi}\int_{\partial\Sigma\times R}\varphi d_- d_+\varphi\right) \\ &\propto \int_i^f \mathcal{D}[\varphi|_{\partial\Sigma\times R}]\delta(\partial_-\varphi)\exp\left(\frac{i}{\hbar}\frac{\nu}{8\pi}\int_{\partial\Sigma\times R}dx^0dx^1(\partial_\mu\varphi)(\partial^\mu\varphi)\right) \end{aligned} \tag{5.2.39}$$

where $x^0 := vt$ and $x^1 := \theta$. Since φ is evaluated only on the edge, the gauge freedom of a (5.2.35) is not present in the path integral over φ.

We see that the dynamics of the gauge field a in the gauge $a_- = 0$ can be completely described by a left-handed scalar field on $\partial\Sigma \times R$ with action

$$S[\varphi] = \frac{\nu}{8\pi} \int_{\partial\Sigma\times R} d^2x (\partial_\mu \varphi)(\partial^\mu \varphi). \tag{5.2.40}$$

If we had chosen the gauge fixing constraint $a_+ = 0$ we would have obtained a right-handed scalar with an action of the form (5.2.40), but with $-\nu$ instead of ν. Positiveness of energy dictates that the factor in front of the integral is positive. If we want to describe a in terms of a well-behaved boson, we have to choose the gauge fixing that makes this factor positive. The sign of ν determines the chirality of the boson field, and consequently of the edge currents, as follows:

$$\begin{aligned} \nu > 0 \quad &\rightarrow \quad \varphi = \varphi_+(x^+) \\ \nu < 0 \quad &\rightarrow \quad \varphi = \varphi_-(x^-). \end{aligned} \tag{5.2.41}$$

5.2.3 Incompressibility

The quantization of a system with only first class constraints is easy. (If there are also second class constraints, one has to define a new bracket, called Dirac bracket, which makes all constraints 'first class' with respect to this bracket.)

One can 'choose' one of the following equivalent approaches:

- Construct a Hilbert space that contains only the non-constrained variables, i.e. only the actual degrees of freedom.
- Allow all variables into Hilbert space and implement the constraints by defining *physical* states |phys> as those that satisfy

$$\hat{F}_i |phys>= 0 \qquad \forall i. \tag{5.2.42}$$

Quantum states that violate a constraint are inherently impossible in the first approach and unphysical in the second one.

Using (5.2.1), we find that the constraint (5.2.18) allows no quantum deviations whatsoever from the equation of motion

$$j^0 = \sigma_H B \tag{5.2.43}$$

except at the very edge of Σ. The fact that the electromagnetic field is massive in Chern-Simons theory means that it can be treated as a classical field at nonmicroscopic distance scales. As the classical B is constant in time and space, it follows that (non-microscopically) the charge density is constrained to be constant. This is an important result; Equation (5.1.3) showed incompressibility at the *classical* level. Now we see that the incompressibility is present also at the *quantum* level.

5.3 Conformal field theory in 1+1 dimensions

The action (5.2.40) of the massless edge boson is conformally invariant. The importance and physical interpretation of this conformal symmetry will become clear in the next chapter. It will turn out that area-preserving deformations of the incompressible quantum fluid have the effect of inducing conformal transformations on the edge of the fluid.

We now present an elementary review of conformal field theory on a cylinder. We discuss the role of the stress-energy tensor as generator of conformal transformations and introduce the Virasoro algebra. Then we quantize a massless fermion and the massless boson describing the QHE at filling fraction ν, and explicitly derive the form of the $U(1)$ current and Virasoro generators in terms of creation/annihilation operators. The current operators are shown to satisfy the $U(1)$ Kač-Moody algebra found in (5.2.24). Then we show the equivalence between bosons and fermions, using the Sugawara construction.

Finally we construct lowest weight representations of the Virasoro and Kač-Moody algebra. It turns out that the case of one boson can only describe the Laughlin fractions $\frac{1}{2p+1}$. The lowest weight representations are found to correspond to composite fermions and anyons. The simple fermion is not an element of the particle spectrum. The Laughlin wave functions are obtained as N-point correlation functions of composite fermion vertex operators.

5.3.1 The stress-energy tensor and the conformal algebra

A theory is called 'conformal' if its action is invariant under the so-called conformal transformations

$$g_{\mu\nu}(x) \to \phi(x) g_{\mu\nu}(x), \tag{5.3.1}$$

i.e. local rescalings of the metric. Conformal invariance is of great importance in the study of phase transitions in statistical mechanics and in string theory. A veritable avalanche of research in conformal field theories was started in 1984 by a paper of Belavin, Polyakov and Zamolodchikov [17]. Most of what follows here can be found in that paper.

An important property of conformal field theories (CFT's) is that the stress-energy tensor is traceless. The action has the form $S = \int d^2x \sqrt{g}\mathcal{L}(x)$, with $g := -\det(g_{\mu\nu})$ and L(x) the Lagrange density. The stress-energy tensor is defined as

$$T^{\mu\nu}(x) := -\frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}(x)}. \tag{5.3.2}$$

For a conformal transformation $\delta g_{\mu\nu}(x) = \epsilon(x) g_{\mu\nu}(x)$ we have

$$\delta S = \int d^2x \frac{\delta S}{\delta g_{\mu\nu}(x)} \delta g_{\mu\nu}(x) = \int d^2x \sqrt{g} T^{\mu}_{\mu}(x) \epsilon(x). \tag{5.3.3}$$

If $\delta S = 0$ then $T^{\mu}_{\mu}(x) = 0 \ \ \forall x$, since $\epsilon(x)$ is an arbitrary function.

In 1+1-dimensional Minkowski space with light cone coordinates $x^{\pm} := x^0 \pm x^1$ the metric is given by

$$ds^2 = dx^- dx^+. \tag{5.3.4}$$

Conformal transformations now take the form

$$x^- \to f^-(x^-); \quad x^+ \to f^+(x^+); \quad ds^2 \to \frac{df^-}{dx^-} \frac{df^+}{dx^+} ds^2, \tag{5.3.5}$$

i.e. they are independent redefinitions of x^- and x^+. Each of these redefinitions is generated by an infinite number of generators, so the conformal symmetry algebra consists of two independent infinite sets of generators. A basis can be chosen as follows

$$V_n^{\pm} := -ie^{-inx^{\pm}} \partial_{\pm}, \tag{5.3.6}$$

with $\partial_\pm := \frac{1}{2}(\partial_0 \pm \partial_1)$. The commutation relations are

$$[V_n^\pm, V_m^\pm] = (n-m)V_{n+m}^\pm \quad ; \quad [V_n^\pm, V_m^\mp] = 0. \tag{5.3.7}$$

These two commuting algebra's are called the classical Virasoro algebras. Note that the Hamiltonian and the momentum can be written as

$$H \propto i\partial_0 = -(V_0^+ + V_0^-) \quad ; \quad P \propto -i\partial_1 = V_0^+ - V_0^- \tag{5.3.8}$$

and that $V_n^{\pm\dagger} = V_{-n}^\pm$.

The conserved Noether charge generating $x^\pm \to x^\pm + \varepsilon^\pm(x^\pm)$ is

$$Q[\varepsilon^+, \varepsilon^-] = \oint dx^1[\varepsilon^+ T_{++} + \varepsilon^- T_{--}] \tag{5.3.9}$$

with

$$\begin{aligned} T_{--} &:= \frac{1}{4}(T_{00} + T_{11} - T_{01} - T_{10}) \\ T_{++} &:= \frac{1}{4}(T_{00} + T_{11} + T_{01} + T_{10}). \end{aligned} \tag{5.3.10}$$

Using the tracelessness $T_\mu^\mu = 0$, the current conservation $\partial_\mu T^{\mu\nu} = 0$ and the fact that $T_{\mu\nu} = T_{\nu\mu}$, we find

$$\begin{aligned} \partial_+ T_{--} = 0 &\implies T_{--} = T_{--}(x^-) \\ \partial_- T_{++} = 0 &\implies T_{++} = T_{++}(x^+). \end{aligned} \tag{5.3.11}$$

The conformal Noether charge (5.3.9) can now be written as

$$Q[\varepsilon^+, \varepsilon^-] = \frac{1}{2}\oint dx^+ \varepsilon^+ T_{++} - \frac{1}{2}\oint dx^- \varepsilon^- T_{--}. \tag{5.3.12}$$

There is an infinite freedom in the choice of ε^+ and ε^-, so there are infinitely many conserved charges.

In quantised theories an important generalisation of (5.3.7) can occur due to the fact that states in Hilbert space are defined only up to a phase factor. In the commutation relation (5.3.7) an extra constant, depending on n and m, may therefore appear. After a simple calculation it turns out that there exists a basis of the algebra in which this constant has the form $c_{nm} = \frac{c}{12}n(n^2-1)\delta_{n+m,0}$, yielding the following algebra for the quantum Virasoro operators L_n,

$$\begin{aligned} [L_n^\pm, L_m^\pm] &= (n-m)L_{n+m}^\pm + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\ [L_n^\pm, L_m^\mp] &= 0. \end{aligned} \tag{5.3.13}$$

The real constant c is called the central extension. The value of c is determined by the specific quantum theory one is dealing with, and thus c is a physical parameter characterising the theory.

The conserved charges (5.3.12) generate conformal transformations. It is convenient to choose a basis consisting of the Laurent coefficients of the stress-energy operator. The Laurent expansion of the stress-energy can be written as

$$\begin{aligned} T_\pm(x^\pm) &=: \pm\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} L_n^\pm e^{inx^\pm}, \\ L_n^\pm &= \pm\oint dx^\pm e^{-inx^\pm} T_\pm(x^\pm), \end{aligned} \tag{5.3.14}$$

where we have introduced the notation $T_- := T_{--}$, $T_+ := T_{++}$. According to (5.3.12), $L_n^\pm$ generates a transformation on Hilbert space that corresponds to the coordinate transformation $x^\pm \to x^\pm + e^{-inx^\pm}$, which is exactly the coordinate transformation generated by the classical V_n (5.3.6). The L_n satisfy the full Virasoro algebra (5.3.13).

Note that the hermiticity of T, $T_\pm^\dagger = T_\pm$, requires $L_n^{\pm\dagger} = L_{-n}^\pm$.

The Hamiltonian and the momentum can again be expressed in terms of L_0. Using the definition of $T_\pm$ (5.3.10) and (5.3.14) we get

$$H = L_0^+ + L_0^- \quad ; \quad P = L_0^+ - L_0^- . \tag{5.3.15}$$

5.3.2 The 1+1 dimensional Dirac theory

The Dirac equation for a free massive fermion in 1+1 dimensions is

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{5.3.16}$$

with ψ a two-component spinor and $\mu = 0, 1$. The corresponding action is

$$S = \int d^2x \left[\frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{i}{2} (\partial_\mu \bar{\psi}) \gamma^\mu \psi - m\bar{\psi}\psi \right] \tag{5.3.17}$$

The γ-matrices are 2×2 matrices defined by their anticommutators,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{5.3.18}$$

The Dirac conjugate $\bar{\psi}$ is defined as

$$\bar{\psi} := \psi^\dagger \gamma^0 \tag{5.3.19}$$

The action (5.3.17) is invariant under the following global gauge transformation

$$\psi \to e^{i\alpha}\psi, \quad \bar{\psi} \to e^{-i\alpha}\bar{\psi}. \tag{5.3.20}$$

The conserved Noether current is the fermion current,

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{5.3.21}$$

and the conserved charge is the total fermion number,

$$N = \int dx^1 j^0 = \int dx^1 \psi^\dagger \psi. \tag{5.3.22}$$

The stress-energy tensor of the free fermion is

$$\begin{aligned} T^\mu{}_\kappa &:= \frac{\partial L}{\partial(\partial_\mu \psi)} \partial_\kappa \psi + \partial_\kappa \bar{\psi} \frac{\partial L}{\partial(\partial_\mu \bar{\psi})} - g^\mu_\kappa L \\ &= \frac{i}{2} \left[\bar{\psi}\gamma^\mu \partial_\kappa \psi - \partial_\kappa \bar{\psi} \gamma^\mu \psi - g^\mu_\kappa (\bar{\psi}\gamma^\lambda \partial_\lambda \psi - \partial_\lambda \bar{\psi} \gamma^\lambda \psi + 2mi\bar{\psi}\psi) \right] \end{aligned} \tag{5.3.23}$$

Using (5.3.16), we find its trace

$$T^\mu{}_\mu = m\bar{\psi}\psi. \tag{5.3.24}$$

In the so-called chiral representation we have

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \quad \bar{\psi} = \left(-\psi_+^\dagger, -\psi_-^\dagger\right) \tag{5.3.25}$$

In the chiral representation, the Lagrange density is

$$\begin{aligned} L &= \frac{i}{2}\left[\psi_-^\dagger(\partial_0+\partial_1)\psi_- - h.c.\right] + \frac{i}{2}\left[\psi_+^\dagger(\partial_0-\partial_1)\psi_+ - h.c.\right] \\ &+ m\left[\psi_-^\dagger\psi_+ + \psi_+^\dagger\psi_-\right] \end{aligned} \tag{5.3.26}$$

We see that a massless fermion can be split into two independent chiral parts ψ_+ and ψ_-, with equations of motion

$$\partial_\mp\psi_\pm = 0. \tag{5.3.27}$$

For a massless fermion, the action (5.3.17) is also invariant under the so-called chiral transformation

$$\psi \to e^{i\alpha\gamma^2}\psi, \qquad \bar{\psi} \to e^{-i\alpha\gamma^2}\bar{\psi}, \tag{5.3.28}$$

and the corresponding chiral Noether current is

$$j^\mu_{chiral} = \bar{\psi}\gamma^\mu\gamma^2\psi. \tag{5.3.29}$$

The conserved charge is the chiral fermion number

$$N_{chiral} = \int dx^1 j^0_{chiral} = \int dx^1 \psi^\dagger\gamma^2\psi. \tag{5.3.30}$$

In the chiral representation, the currents are written as

$$\begin{aligned} j^0 &= \psi_-^\dagger\psi_- + \psi_+^\dagger\psi_+ \quad ; \quad j^1 = \psi_-^\dagger\psi_- - \psi_+^\dagger\psi_+ \\ j^0_{chiral} &= \psi_-^\dagger\psi_- - \psi_+^\dagger\psi_+ \quad ; \quad j^1_{chiral} = \psi_-^\dagger\psi_- + \psi_+^\dagger\psi_+ \\ j^\pm &:= \frac{1}{2}(j_0 \pm j_1) = \psi_\pm^\dagger\psi_\pm \end{aligned} \tag{5.3.31}$$

The stress-energy tensor of a massless fermion is traceless and the action is conformally invariant. The conserved Noether currents $T_\pm$ are

$$T_\pm = \frac{i}{2}(\psi_\pm^\dagger\partial_\pm\psi_\pm - h.c). \tag{5.3.32}$$

The chiral fields $\psi_\pm$ can be quantized in the following way

$$\psi_\pm(x^\pm) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} a_n^\pm e^{i(n-\mu)x^\pm}, \tag{5.3.33}$$

The normalization is chosen such that the canonical equal-time anticommutation relations

$$\{\psi_\pm^\dagger(x), \psi_\pm(x')\} = \delta(x-x') \tag{5.3.34}$$

are satisfied iff the creation and annihilation operators $a_k^{\pm}$ satisfy

$$\{a_n^{\pm}, a_m^{\pm\dagger}\} = \delta_{n,m} \tag{5.3.35}$$

with all other anticommutations zero. The factor y^{μ} contains information about the spatial boundary conditions:

$$\psi_{\pm}(x^1 + 2\pi) = e^{\pm i2\pi\mu}\psi_{\pm}(x^1). \tag{5.3.36}$$

The parameter μ can only take the values $\mu = 0$ ('Ramond') or $\mu = \frac{1}{2}$ ('Neveu-Schwarz'), corresponding to periodic or antiperiodic boundary conditions. We will choose $\mu = \frac{1}{2}$, since we are dealing with a fermion.

The vacuum $|0>$ is defined by the relations

$$\begin{aligned} a_n^{\pm}|0> &= 0 \quad , \quad n > 0 \\ a_n^{\pm\dagger}|0> &= 0 \quad , \quad n < 0. \end{aligned} \tag{5.3.37}$$

With this quantization of ψ, the Virasoro generators (5.3.14) and the current $j^{\pm}(y_{\pm}) = \frac{1}{2\pi}\sum j_n^{\pm} y_{\pm}^{-n}$ take the form

$$\begin{aligned} L_n^{\pm} &= \sum_{p=-\infty}^{\infty} (p - \mu - \frac{n}{2}) a_{p-n}^{\pm\dagger} a_p^{\pm} \\ j_n^{\pm} &= \sum_{p=-\infty}^{\infty} a_{p-n}^{\pm\dagger} a_p^{\pm}. \end{aligned} \tag{5.3.38}$$

However, $L_0^{\pm}$ and $j_0^{\pm}$ are infinite. This can be cured by normal ordering, placing all creation operators left of the annihilationoperators. The ordering is in fact a subtraction of the infinite ground state energy, so one is free to include an arbitrary finite constant in $L_0^{\pm}$. The conventional choice leads to the following algebra of normal ordered operators

$$[L_n^{\pm}, L_m^{\pm}] = (n-m)L_{n+m}^{\pm} + \frac{1}{12}(n^3 - n)\delta_{n+m,0} \tag{5.3.39}$$

$$[j_n^{\pm}, j_m^{\pm}] = m\delta_{n+m,0} \tag{5.3.40}$$

$$[L_n^{\pm}, j_m^{\pm}] = -m j_{n+m}^{\pm}. \tag{5.3.41}$$

The first equation is the Virasoro algebra with central extension $c = 1$. The second one is a $U(1)$ Kač-Moody algebra.

5.3.3 The massless scalar field

The action of the massless edge boson that we found from the pure Chern-Simons description of the QHE was

$$S_{CS}[\varphi] = \frac{|\nu|}{8\pi}\int d^2x(\partial_{\mu}\varphi)(\partial^{\mu}\varphi). \tag{5.3.42}$$

The stress-energy tensor is

$$T^{\mu}_{\lambda} = \frac{|\nu|}{4\pi}\left[(\partial^{\mu}\varphi)(\partial_{\lambda}\varphi) - \frac{1}{2}g^{\mu}_{\lambda}(\partial_{\kappa}\varphi)(\partial^{\kappa}\varphi)\right]. \tag{5.3.43}$$

The trace $T^{\mu}{}_{\mu}$ is zero, so the action of the massless boson is conformally invariant. The conformal Noether currents $T_{\pm}$ are

$$T_{\pm} = \frac{|\nu|}{16\pi}(\partial_{\pm}\varphi)(\partial_{\pm}\varphi). \tag{5.3.44}$$

The action (5.3.42) is also invariant under the transformation

$$\varphi \to \varphi + \Lambda \quad ; \quad \partial_\mu \partial^\mu \Lambda = 0. \tag{5.3.45}$$

This is in fact nothing but a gauge transformation of the $U(1)$-field g (5.2.37), $g \to e^{-i\Lambda} g$. The conserved charge is

$$Q[\Lambda] = \frac{|\nu|}{2\pi} \oint_{\partial\Sigma} dx^1 [\Lambda \partial_0 \varphi - \frac{1}{2} \partial_0 (\Lambda \varphi)]. \tag{5.3.46}$$

If we consider a chiral field $\varphi = \varphi_\pm$, the charge is written as

$$Q^\pm[\Lambda] = \frac{|\nu|}{2\pi} \oint_{\partial\Sigma} dx^1 \Lambda \partial_\pm \varphi =: \oint_{\partial\Sigma} dx^1 \Lambda \mathcal{J}^\pm. \tag{5.3.47}$$

The quantity $\mathcal{J}^\pm$ is the chiral edge current.

The chiral fields can be quantized as follows

$$\varphi_\pm(x^\pm) = q^\pm + \frac{1}{|\nu|} p^\pm x^\pm + \frac{i}{\sqrt{|\nu|}} \sum_{n\neq 0} \frac{b_n^\pm}{n} e^{inx^\pm}, \tag{5.3.48}$$

where q and p are hermitian operators and $b_n^{\pm\dagger} = b_{-n}^\pm$. The canonical commutation

$$[\frac{|\nu|}{4\pi} \partial_0 \varphi(x^0, x^1), \varphi(x^0, x'^1)] = -i\delta(x^1 - x'^1) \tag{5.3.49}$$

is satisfied iff the only nonvanishing commutations are

$$\begin{aligned} [p^\pm, q^\pm] &= -i \\ [b_n^\pm, b_m^\pm] &= m\delta_{n+m,0} \end{aligned} \tag{5.3.50}$$

with all other commutators zero. The vacuum is defined by the relation

$$b_n^\pm |0>= 0 \qquad n \geq 0 \tag{5.3.51}$$

where $b_0^\pm$ is defined as $-p^\pm/\sqrt{|\nu|}$.

With this quantization, the conformal charges $L_n^\pm$ and the Fourier modes $\mathcal{J}_n^\pm$ of the current become

$$\begin{aligned} L_n^\pm &= \frac{1}{2} \sum_{k=-\infty}^{\infty} b_{n-k}^\pm b_k^\pm \\ \mathcal{J}_n^\pm &= \sqrt{|\nu|} b_n^\pm. \end{aligned} \tag{5.3.52}$$

As was the case with the fermion, L_0 and $\mathcal{J}_0$ are divergent and all operators have to be normal ordered by placing all creation operators to the left of the annihilation operators. The commutations of the normal ordered generators are

$$[L_n^\pm, L_m^\pm] = (n-m) L_{n+m}^\pm + \frac{1}{12}(n^3 - n)\delta_{n+m,0} \tag{5.3.53}$$

$$[\mathcal{J}_n^\pm, \mathcal{J}_m^\pm] = |\nu| m \delta_{n+m,0} \tag{5.3.54}$$

$$[L_n^\pm, \mathcal{J}_m^\pm] = -m \mathcal{J}_{n+m}^\pm. \tag{5.3.55}$$

The first one is the Virasoro algebra with $c = 1$. The second relation is a $U(1)$ Kač-Moody algebra of level $|\nu|$.

The charges $Q^{\pm}[\Lambda]$ have commutation relations that are equivalent to the Poisson bracket (5.2.24) of the observables of the classical edge theory,

$$-i\left[Q^{\pm}[\Lambda_1], Q^{\pm}[\Lambda_2]\right] = \pm\frac{|\nu|}{2\pi}\oint_{\partial\Sigma}\Lambda_1 d\Lambda_2. \tag{5.3.56}$$

Again we have to choose the chirality of the current as in (5.2.41) in order to correctly describe the QHE.

Note that the factor $|\nu|$ in (5.3.54) differs from the fermionic case (5.3.40). Apparently, the edge current is not given by the fermion current $j^{\pm}$. We will see in a moment that this is because the relevant particles are *composite* fermions.

5.3.4 Fermion-boson equivalence

The fact that the central charge is the same for a fermion and a boson is no coincidence: In 1+1 dimensions both theories are completely equivalent and can be written in terms of each other.

The chiral current conservation $\partial_\mu j^\mu_{chiral} = 0$ allows us to write

$$j^\mu_{chiral} = \alpha\varepsilon^{\mu\kappa}\partial_\kappa\varphi \tag{5.3.57}$$

where φ is some real scalar field and α a real constant. The fermion current conservation $\partial_\mu j^\mu = 0$ and the relation $j^\mu_{chiral} = -\varepsilon^{\mu\lambda}j_\lambda$ show that φ satisfies a massless Klein-Gordon equation

$$\partial_\mu\partial^\mu\varphi = 0. \tag{5.3.58}$$

If φ satisfies the canonical commutation relation (5.3.49) then the requirement that we obtain the Kač-Moody algebra (5.3.40) fixes the proportionality factor α in (5.3.57) to $\alpha = -\sqrt{|\nu|}$, giving

$$\begin{aligned} j^\mu &= \sqrt{|\nu|}\partial^\mu\varphi \\ j^{\pm}_n &= b^{\pm}_n. \end{aligned} \tag{5.3.59}$$

We are now able to express ψ in terms of φ. To this end we use the Sugawara form [18] of the fermionic stress-energy tensor,

$$T^{\mu\lambda}_{Sug} = \frac{1}{2} : (j^\mu j^\lambda + j^\lambda j^\mu - g^{\mu\lambda}j_\kappa j^\kappa) : \tag{5.3.60}$$

By substitution of (5.3.59) into this expression it is readily seen that it corresponds to the bosonic stress-energy (5.3.43),

$$T^{\mu\lambda}_{Sug} = 4\pi T^{\mu\lambda}_{boson}. \tag{5.3.61}$$

The normal ordering in (5.3.60) is defined with respect to the the current operators $j^{\pm}_n$.

Using the Heisenberg equation

$$\partial_\mu A = i[P_\mu, A], \tag{5.3.62}$$

where A is an arbitrary operator and P_μ is the total momentum of the fermion, $P^\mu = \oint dx^1 T^{0\mu}_{Sug}$, we get the following relation between $\psi_\pm$ and $j^{\pm}$,

$$\partial_1\psi_\pm = \mp i j^{\pm}\psi_\pm. \tag{5.3.63}$$

Formal integration finally gives

$$\begin{aligned}\psi_\pm(x^0 \pm x^1) &\propto \; : e^{\mp i\sqrt{|\nu|}\int^{x^1} dx^{1'} \partial_\pm \varphi_\pm (x^0 \pm x^{1'})} : \\ &\propto \; : e^{-i\sqrt{|\nu|}\varphi_\pm(x^0 \pm x^1)} : \end{aligned} \quad (5.3.64)$$

5.3.5 Representation theory of the Kač-Moody algebra

When the action of a physical system has a symmetry, the Hilbert space forms a representation (or tensor product of several representations) of the symmetry algebra. The Hilbert space of our massless chiral boson system decomposes into a number of lowest weight representations of the Kač-Moody algebra or the Virasoro algebra. Every lowest weight corresponds to a particle.

A lowest weight state $|Q^+, Q^- >$ is defined as an eigenstate of the energy operator $L_0^\pm$ and the charge operator $\mathcal{J}_0^\pm$, that is annihilated by all step-down operators.

$$\begin{aligned}\mathcal{J}_0^\pm |Q^+, Q^- > &= Q^\pm |Q^+, Q^- > \\ L_0^\pm |Q^+, Q^- > &= \frac{(Q^\pm)^2}{2|\nu|} |Q^+, Q^- > \end{aligned} \quad (5.3.65)$$

The eigenvalues $Q^\pm$ are called lowest weights (or charges). The eigenvalues of $L_0^\pm$ are called conformal weights. It is evident from (5.3.39) and (5.3.54) that the Virasoro generators and the current modes are laddering operators that ladder the $L_0^\pm$-eigenvalues. The charge is not affected. The modes with positive n are step-down operators. The lowest weight condition reads

$$\begin{aligned} L_n^\pm |Q^+, Q^- > = 0, \quad n > 0 \\ \mathcal{J}_n^\pm |Q^+, Q^- > = 0, \quad n > 0. \end{aligned} \quad (5.3.66)$$

It is clear that the vacuum $|0>$ is a lowest weight state with zero energy and charge. A complete representation is created from a lowest weight state by acting on it with all possible combinations of the step-up operators $\mathcal{J}_{-n}^\pm$ $(n > 0)$. (Alternatively, one can choose to act with all the $L_{-n}^\pm$, $n > 0$. The result is the same.) The resulting representation is labeled by the weights $Q^\pm$.

In the edge theory of the QHE the fields are chiral, so we will only be concerned with representations where either Q^+ or Q^- is zero. The lowest weight states will be denoted as $|Q>_\pm$

It is possible to go from one representation to another using a so-called *vertex operator*. This is an operator of the form

$$V_k^\pm(x^\pm) = \; : e^{-ik\varphi_\pm(x^\pm)} : \quad (5.3.67)$$

When it acts on a state it alters the charge by an amount k, thereby changing the weight,

$$[\mathcal{J}_0^\pm, V_k^\pm] = k V_k^\pm. \quad (5.3.68)$$

When acting on the vacuum, $V_k^\pm(x^\pm)$ creates a coherent state at position $x^\pm$. All lowest weight states can be obtained from the vacuum using only the zero-mode of $\varphi_\pm$,

$$|Q>_\pm = e^{-iQq^\pm} |0> . \quad (5.3.69)$$

(The operator $e^{-iQq^{\pm}}$ can be viewed abstractly as the limit of the vertex operator $V_Q^{\pm}(x^{\pm})$ for $e^{-ix^{\pm}} \to 0$. This would correspond to imaginary time going to $-\infty$.)

Every representation with weight Q corresponds to a particle with charge Q. The operator $V_{-Q}^{\pm}$ can be interpreted as the field operator of that particle. In fact, expression (5.3.64) is an example of this identification. However, if φ is compactified then Q is quantized. We require that the electron and the electron hole are elements of the particle spectrum. Their vertex operators $V_{\pm 1}^{\pm}$ have to be single-valued and they should obey standard anticommutation relations. They can only be single-valued if we make the identification

$$\varphi_{\pm} \approx \varphi_{\pm} + 2\pi M \quad , \quad M \in \mathbb{Z}. \tag{5.3.70}$$

(A quantity often encountered in the literature is the so-called *compactification radius* R, which is in our case defined by the relation $\varphi_{\pm} \approx \varphi_{\pm} + \frac{2\pi}{|\nu|} R\mathbb{Z}$. We have the identification $R = |\nu|$.)

Furthermore, φ is also periodic in the spatial variable x^1. Together with the compactification (5.3.70) this gives the condition $\varphi_{\pm}(x^1 + 2\pi) \approx \varphi_{\pm}(x^1)$, which can be written as

$$\varphi_{\pm}(x^1) \mp \frac{2\pi}{|\nu|} \mathcal{J}_0^{\pm} \approx \varphi_{\pm}(x^1). \tag{5.3.71}$$

This condition is satisfied only if the eigenvalues of $\mathcal{J}_0^{\pm}$ are integer multiples of $|\nu|$, i.e.

$$Q = M\nu, \quad M \in \mathbb{Z}. \tag{5.3.72}$$

(The general result in terms of the compactification radius is that the spectrum of $p^{\pm}$-eigenvalues is given by $p^{\pm} = \frac{N}{R} \pm \frac{MR}{2}$, with $M, N \in \mathbb{Z}$. For a chiral field this becomes $p^{\pm} = \pm MR$.)

The requirement that the $V_{\pm 1}^{\pm}$ satisfy anticommutation relations poses serious restrictions on the filling fraction ν. Using the canonical commutations (5.3.50) and the relation

$$e^A e^B = e^{[A,B]} e^B e^A, \tag{5.3.73}$$

which holds if $[A, [A, B]] = [B, [B, A]] = 0$, we find that the exchange behaviour of $V_Q^{\pm}$ depends on ν and that the statistical parameter s is equal to the conformal weight,

$$\begin{aligned} V_Q^{\pm}(x^0 \pm x^1) V_Q^{\pm}(x^0 \pm x^{1'}) &= (-1)^{\frac{Q^2}{|\nu|}} V_Q^{\pm}(x^0 \pm x^{1'}) V_Q^{\pm}(x^0 \pm x^1) \\ s_Q &= \frac{Q^2}{2|\nu|} \end{aligned} \tag{5.3.74}$$

The electron(hole) operator obeys fermionic statistics iff ν^{-1} is an odd integer,

$$|\nu| = \frac{1}{2p+1}. \tag{5.3.75}$$

Apparently, only the Laughlin fractions can be obtained with the ansatz $j = *d\beta$ (see (5.1.30)). In the next section a more general ansatz with several incompressible fluids will be used.

It is now easy to make a classification of all existing edge particles at the Laughlin filling fractions ν_p. Their charge is given by

$$Q_{p,M} = \frac{M}{2p+1} \quad , \quad M \in \mathbb{Z} \tag{5.3.76}$$

Equation (5.3.74) shows that their statistical parameter is

$$s_{p,M} = \frac{1}{2}\frac{M^2}{(2p+1)}. \qquad (5.3.77)$$

These are important results: The electron(hole) has statistics $s_{p,\pm\nu^{-1}} = (2p+1)/2$, which corresponds to a composite fermion in the Jain model, consisting of one electron(hole) and $2p$ flux quanta. The vertex operators with fractional charge $Q = \pm\frac{1}{2p+1}$ are multi-valued and have fractional statistics $s_{p,\pm 1} = \frac{1}{2(2p+1)}$. These are indeed precisely the quasiparticle/hole excitations that arise from the Laughlin ground state.

The rest of the particle spectrum corresponds to multiples of these two basic particle types. (In fact the composite fermion is a combination of $2p+1$ quasiparticles.) Note that the bare electron (5.3.64) is *not* a part of the spectrum (unless $|\nu| = 1$).

5.3.6 Reobtaining the Laughlin wave functions

Since the electron field has the form of a vertex operator, many-electron wave functions are expressed as N-point correlation functions of vertex operators. Given that there are N_e electrons, the amplitude of finding them at positions $\xi_1, \cdots, \xi_{N_e}$ at time x^0 is obtained by taking the inner product <out|in> of an incoming state with an outgoing state,

$$|out> = \left[\prod_{j=1}^{N_e} V_{-1}^{\pm}(x^0 \pm \xi_j)\right] |0> . \qquad (5.3.78)$$

There are some subtleties in the construction of the inner product. The |in>-state is given by

$$|in> = e^{-iN_e q}|0> . \qquad (5.3.79)$$

It describes a state with charge N_e and constant density. It can be thought of as a series of N_e electron creations at time $x^0 = -\infty$.

The product of vertex operators has to be time ordered, giving

$$\Psi^*(x^0;\xi_1,\cdots,\xi_{N_e}) = < in|out > = < 0|e^{iN_e q}\prod_{j=1}^{N_e} : e^{i\varphi_\pm(x^0\pm\xi_j)} : |0> . \qquad (5.3.80)$$

This expectation value is calculated using the relation

$$< 0|\prod_{j=1}^{M} : e^{iA_j} : |0> = \prod_{j<k}^{M} e^{-<0|A_j A_k|0>}, \qquad (5.3.81)$$

where the A_j are arbitrary operators satisfying $< 0|A_j|0 >= 0$. We are allowed to use equation (5.3.81), because the field φ has zero vacuum expectation value. Using the canonical equal time commutation relations we find

$$< 0|\varphi_\pm(\xi_1)\varphi_\pm(\xi_2)|0> = \frac{1}{|\nu|}\ln(e^{-ix_1^\pm} - e^{-ix_2^\pm}), \qquad (5.3.82)$$

with $x_j^\pm := x^0 \pm \xi_j$. Substitution into (5.3.80) yields

$$\Psi^*(x^0;\xi_1,\cdots,\xi_{N_e}) = \prod_{j<k}^{N_e}(e^{-ix_j^\pm} - e^{-ix_k^\pm})^{2p+1} \propto \prod_{j<k}^{N_e}(e^{\mp i\xi_j} - e^{\mp i\xi_k})^{2p+1}. \qquad (5.3.83)$$

The time dependence has been left out in the second expression, since it gives only an unimportant phase factor. If we choose $\varphi = \varphi_+$ and a disc shape of the sample we get

$$\Psi(x^0; \xi_1, \cdots, \xi_{N_e}) \propto \prod_{j<k}^{N_e} (z_j - z_k)^{2p+1}, \tag{5.3.84}$$

where z is defined as $R_0 e^{i\xi}$, with R_0 the radius of the disc. Apart from the exponential factor $\exp(-\frac{1}{2}\sum_j |z_j|^2)$, this is the Laughlin wave function ψ_p on the edge of the sample. It gives the probability amplitude of finding N_e electrons at positions $z = R_0 e^{i\xi_j}$, given that they are located on the edge. Taking the analytic continuation of Ψ into the disc, we get the analytic part of the Laughlin wave function for the whole sample.

5.4 More than a single Hall fluid

We have seen that the ansatz $j = *d\beta$ (5.1.30) only yields the Laughlin fractions. The logical next step is to try an approach with more than one incompressible fluid in the hope of obtaining the other Hall fractions. Following Fröhlich and Zee [22], we generalize the pure Chern-Simons action $S[a] = \int a \wedge da$ (5.2.2) for the case of several gauge fields a_I. The symmetry of the generalized theory is $\widehat{U(1)}^{\otimes m}$, one Kač-Moody algebra per Landau level. The particle spectrum is a representation of this algebra. In a certain way we get an m-fold repetition of the one fluid case; Every Landau level has its own type of composite fermion. However, the charges and statistics of the quasiparticles are different.

In the m-fluid model, Jain's composite fermion hierarchy is reobtained in a natural way.

5.4.1 The generalised model

The simplest way of generalizing $j = *d\beta$ is to assume that the electric current of electrons belonging to each Landau level is *separately* conserved. This is not an unrealistic assumption, since the energy gap between separate levels is very large in a strong magnetic field. Repetition of the arguments in sections 5.1 and 5.2 leads to m potentials for the current when m Landau levels contain electrons,

$$j = \sum_{I=1}^{m} j^I = \sum_{I=1}^{m} *d\beta^I. \tag{5.4.1}$$

Every Landau level has its own Hall conductivity σ^I,

$$d\beta^I = -\sigma^I dA. \tag{5.4.2}$$

The total Hall conductivity is given by the sum of the separate contributions

$$\sigma_H = \sum_{I=1}^{m} \sigma^I. \tag{5.4.3}$$

The equation of motion (5.4.2) can be obtained from a Chern-Simons action of the general form

$$S[A, \beta^1, \cdots, \beta^m] = -\frac{h}{2e^2} \sum_{I,J}^{m} K_{IJ} \int (\beta^I + \sigma^I A) \wedge d(\beta^J + \sigma^J A), \tag{5.4.4}$$

where K is some real-valued dimensionless symmetric invertible $m \times m$ matrix. Its physical interpretation is that the K_{IJ} are current-current or current-gauge field coupling parameters. K contains information about the statistical properties of the current carriers in every Landau level.

The filling fraction is encoded in K as follows: The factor in front of the term $\int A \wedge dA$ is equal to $-\sigma_H/2$ as in equation (5.1.32),

$$\frac{h}{2e^2} K_{IJ} \sigma^I \sigma^J = \frac{\sigma_H}{2}. \tag{5.4.5}$$

Using (5.4.3) we find

$$\nu = \sum_{I,J}^{m} (K^{-1})^{IJ}. \tag{5.4.6}$$

We need physical arguments in order to determine what K should be. Let us first find out what kind of edge theory is defined by our action (5.4.4). We can rewrite it in a form similar to (5.2.2),

$$\hbar^{-1} S[A, a^1, \cdots, a^m] = -\frac{K_{IJ}}{4\pi} \int (\frac{h}{e^2} \sigma^I a^I) \wedge d(\frac{h}{e^2} \sigma^J a^J) \tag{5.4.7}$$

where a^I is defined as in (5.2.1), but with an extra index. Repetition of the arguments in section 5.2 gives us one massless chiral boson for every Landau level. We have

$$\frac{h}{e^2} \sigma^I a^I = d\varphi^I. \tag{5.4.8}$$

Substitution into (5.4.7) gives

$$\hbar^{-1} S[\varphi^1, \cdots, \varphi^m] = \frac{K_{IJ}}{8\pi} \int_{\partial\Sigma \times R} d^2x (\partial_\mu \varphi^I)(\partial^\mu \varphi^J). \tag{5.4.9}$$

The equations of motion following from this action are just massless Klein-Gordon equations for every boson, $\partial_\mu \partial^\mu \varphi^I = 0$. The symmetry algebra of the action (5.4.9) is $\widehat{U(1)}^{\otimes m}$, i.e. the tensor product of m Kač-Moody algebras; The K-matrix can be diagonalised, since it is symmetric and invertible. In a basis in which K is diagonal, (5.4.4) becomes the action of m decoupled scalar fields γ_a. The fields γ_a can be scaled in such a way that the action is

$$\hbar^{-1} S[\gamma_1, \cdots, \gamma_m] = -\frac{1}{4\pi} \sum_{a=1}^{m} \kappa_a \int d^2x \ (\partial_1 \gamma_a)(\partial_0 + \kappa_a \partial_1)\gamma_a, \tag{5.4.10}$$

where the κ_a take values ± 1 according to the chirality of the fields γ_a. [The action 5.4.10 is equivalent to an action with the standard terms $\sum_a \kappa_a (\partial_\mu \gamma_a)(\partial^\mu \gamma_a)$ combined with the constraint $(\partial_0 - \kappa_a \partial_1)\gamma_a = 0$.]

The physical fields φ^I are expressed in terms of the decoupled fields γ_a using a set of basis vectors $\vec{v}^I$,

$$\begin{aligned} \varphi^I &= \vec{v}^I \cdot \vec{\gamma}, \\ \vec{\gamma} &= K_{IL} \vec{v}^I \varphi^L. \end{aligned} \tag{5.4.11}$$

where the dot product is defined as

$$\vec{v}^I \cdot \vec{\gamma} := \sum_{a=1}^{m} \kappa_a (\vec{v}^I)_a \gamma_a. \tag{5.4.12}$$

The basis vectors sastisfy the following relation

$$\vec{v}^I \cdot \vec{v}^J = (K^{-1})^{IJ}. \tag{5.4.13}$$

The decoupled fields γ_α will play an important role in the next chapter. For now we will concentrate on the physical fields.

The fields φ^I can be quantized as follows,

$$\varphi^I_\pm(x^\pm) = q^I_\pm \pm (K^{-1})^{IJ} p^\pm_J x^\pm + i \sum_{n\neq 0} \frac{b^{I\pm}_n}{n} e^{inx^\pm}. \tag{5.4.14}$$

The canonical equal-time commutation relation $[\Pi_I(x^1), \varphi^J(x^{1'})] = -i\delta^J_I \delta(x^1 - x^{1'})$ is satisfied iff

$$\begin{aligned} [p^\pm_I, q^J_\pm] &= -i\delta^J_I \\ [b^{I\pm}_n, b^{J\pm}_l] &= \pm(K^{-1})^{IJ} l \delta_{n+l,0}. \end{aligned} \tag{5.4.15}$$

Using (5.4.8) and (5.4.1) we find that the current $\mathcal{J}^I$ in the I'th Landau level is given by

$$\mathcal{J}^{I\pm}_n = b^{I\pm}_n, \tag{5.4.16}$$

(the identification $b^{I\pm}_0 = \mp(K^{-1})^{IJ} p^\pm_J$ has been made) with commutation relations

$$[\mathcal{J}^{I\pm}_n, \mathcal{J}^{J\pm}_l] = \pm(K^{-1})^{IJ} l \delta_{n+l,0}. \tag{5.4.17}$$

The construction of lowest weight representations is essentially the same as in the previous section, but now everything is repeated m times. The irreducible representations of the symmetry algebra again correspond to particles.

The total charge in the I'th Landau level is given by $\mathcal{J}^{I\pm}_0 = \mp(K^{-1})^{IJ} p^\pm_J$. A lowest weight state is an eigenstate of the $\mathcal{J}^{I\pm}_0$ that is annihilated by all the positive current modes,

$$\begin{aligned} |Q^1, \cdots, Q^m> &= e^{\mp i Q^I K_{IJ} q^J_\pm}|0> \\ \mathcal{J}^{I\pm}_0|Q^1, \cdots, Q^m> &= Q^I|Q^1, \cdots, Q^m> \\ \mathcal{J}^{I\pm}_n|Q^1, \cdots, Q^m> &= 0, \qquad n > 0. \end{aligned} \tag{5.4.18}$$

The vertex operators creating a set of particles of charge Q^I at position x is given by

$$V^\pm_{\vec{Q}}(x) = \, : e^{\mp i Q^I K_{IJ} \varphi^J_\pm(x)} : \tag{5.4.19}$$

Using the commutation relations (5.4.15) we find the following exchange behaviour,

$$\begin{aligned} V^\pm_{\vec{Q}}(x') V^\pm_{\vec{Q}}(x) &= e^{i\theta_{\vec{Q}}} V^\pm_{\vec{Q}}(x') V^\pm_{\vec{Q}}(x) \\ \frac{\theta_{\vec{Q}}}{\pi} &= K_{IJ} Q^I Q^J. \end{aligned} \tag{5.4.20}$$

We require that the particle spectrum contains a fermionic particle of charge ± 1 in every Landau level, representing an electron(hole) or a composite fermion. Furthermore, an even(odd) number of these particles should yield an exchange phase factor of $e^{i\theta} = +1(-1)$. This gives the following constraint on K:

$$K_{IJ} = integer \quad ; \quad K_{II} = odd. \tag{5.4.21}$$

The compactification of the boson fields, $\varphi^I_\pm \approx \varphi^I_\pm + 2\pi$, dictates the rest of the particle spectrum. The $p^\pm_I$ have integer eigenvalues, so we find

$$\begin{aligned} Q^I &= (K^{-1})^{IJ} n_J, \\ \frac{\theta_{\vec{Q}}}{\pi} &= K_{IJ} Q^I Q^J = (K^{-1})^{IJ} n_I n_J, \quad n_K \in \mathbb{Z}. \end{aligned} \tag{5.4.22}$$

Apart from the (composite) fermions that we imposed, we find particles with fractional charge, since the elements of K^{-1} are fractions. The statistics of these particles are also fractional.

In the basis of the decoupled fields, the lowest weight states $|Q^1, \cdots, Q^m>$ can be written as eigenstates of the zero modes of the γ_α that are annihilated by the positive modes. The weights in this basis are given by a set $\vec{r} := \{r_1, \cdots, r_m\}$ satisfying

$$\vec{r} = \sum_{L=1}^{m} n_L \vec{v}^L, \quad n_L \in \mathbb{Z} \tag{5.4.23}$$

so that the (basis invariant) charge and statistics are given by (5.4.22):

$$\begin{aligned} \mathcal{J}^I_0 |\vec{r}> &= \vec{v}^I \cdot \vec{r}\, |\vec{r}> = (K^{-1})^{IJ} n_J |\vec{r}> \\ \frac{\theta_{\vec{r}}}{\pi} &= K_{IJ} Q^I Q^J = (K^{-1})^{IJ} n_I n_J. \end{aligned} \tag{5.4.24}$$

The set of physically allowed weight vectors $\vec{r}$ forms a lattice $\Gamma_{\{\vec{v}\}}$, spanned by the basis vectors $\{\vec{v}^1, \cdots, \vec{v}^m\}$, with K^{-1} the metric of the lattice and K satisfying (5.4.21). In the construction of a full Hilbert space, the set of lowest weight states has to be closed with respect to the *fusion rules*; these are rules for combining two lowest weight representations. For the algebra $\widehat{U(1)}^{\otimes m}$ the fusion rule is a simple addition of weight vectors,

$$|\vec{r}> \times |\vec{r'}> = |\vec{r} + \vec{r'}> . \tag{5.4.25}$$

The lattice $\Gamma_{\{\vec{v}\}}$ is evidently closed under this fusion rule. These facts will be important in the discussion of $W_{1+\infty}$ representations in the next chapter.

The values of ν that can be obtained from a K-matrix of the form (5.4.21) using (5.4.6) are either integers or fractions with *odd* denominator, since $\det K$ =odd. However, many fractions are found in this way that have not been experimentally observed. Extra arguments are needed to explain why some fractions are more stable than others.

5.4.2 Implementation of Jain's hierarchy

The basic idea, used in Jain's hierarchy, that a state is stable when all electrons have (almost) the same amount of flux, has a natural implementation in the m-fluid model in terms of the current-gauge field interaction. This yields a prescription for K in order to obtain Jain's hierarchy.

The simplest system with m Hall fluids is the integer QHE with m noninteracting completely filled Landau levels. In this case the action takes the simple form

$$S_0 = \pm \frac{h}{2e^2} \sum_{I=1}^{m} \int (\beta^I + A) \wedge d(\beta^I + A) \tag{5.4.26}$$

and we have $K_{IJ} = \pm\delta_{IJ}$.

Fröhlich and Zee [22] reasoned that the only one way in which the electrons in different Landau levels can have interactions with each other is through a Chern-Simons term, since the total action has to be of the form (5.4.4). The gauge field A^μ should couple to the *total* current $\sum_I *d\beta^I$, so the interaction term can only be of the form

$$S_{int} = 2p \cdot \frac{h}{2e^2} \int (\sum_{I=1}^{m} \beta^I + A) \wedge d(\sum_{J=1}^{m} \beta^J + A), \tag{5.4.27}$$

where p is the interaction strength; at this point it could take any real value. Apart from an uninteresting overall minus sign, the K-matrix due to S_0 and S_{int} becomes

$$K_{IJ} = 2p + \alpha\delta_{IJ}, \tag{5.4.28}$$

where p is now taken positive and $\alpha = \pm 1$ accounts for the equal/opposite signs of S_0 and S_{int}. The K-matrix has to be integer, so we have

$$p \in N. \tag{5.4.29}$$

What filling fraction does this situation describe? Inversion of K yields

$$\begin{aligned} (K^{-1})^{IJ} &= \alpha(\delta^{IJ} - \frac{2p}{2pm+\alpha}) \\ \nu &= \sum_{I,J}^{m} (K^{-1})^{IJ} = \frac{m}{2pm+\alpha}. \end{aligned} \tag{5.4.30}$$

This is precisely the first level of Jain's hierarchy. The interaction strength $2p$ between the Landau levels is just the amount of flux that is attached to the electrons(holes) in the composite fermion model.

Using (5.4.22) we find the spectrum of particles and their statistics,

$$\begin{aligned} Q^I &= \alpha(n_I - \frac{2p}{2pm+\alpha}\sum_{J=1}^{m} n_J) \\ \frac{\theta_{\vec{Q}}}{\pi} &= \alpha\left[\sum_{I=1}^{m} n_I^2 - \frac{2p}{2pm+\alpha}(\sum_{I=1}^{m} n_I)^2\right]. \end{aligned} \tag{5.4.31}$$

Apart from the composite fermions, we also have particles with fractional charge and statistics, both with denominator $2pm + \alpha$.

In chapter 4, we saw that the next level of Jain's hierarchy is obtained by deviating a little from the strictly uniform way of attaching flux to the electrons. In the context of the Chern-Simons action (5.4.4), this means a slight relaxation of the requirement that the gauge field A interacts only with the total current.

We add n noninteracting completely filled Landau levels to the m interacting ones. The K-matrix becomes a $(m+n) \times (m+n)$ matrix with a block-diagonal form,

$$\begin{aligned} K &\to K_1 := \begin{pmatrix} K & 0 \\ 0 & \alpha 1\!\!1_{n\times n} \end{pmatrix}, \\ \nu &\to \nu_1 := \sum_{I,J}^{m+n} (K_1^{-1})^{IJ} = n\alpha + \frac{m}{2pm+\alpha}. \end{aligned} \tag{5.4.32}$$

An interaction between the blocks is now turned on. This is again done by coupling A to the total current. The K-matrix now takes the form

$$K_1 \to K_2 \quad ; \quad (K_2)_{IJ} = (K_1)_{IJ} + 2p_2, \tag{5.4.33}$$

where $2p_2$ ($p_2 \in \mathbb{Z}$) is the interaction strength. For the value of the filling fraction of the interacting system we find

$$\begin{aligned} (K_2^{-1})^{IJ} &= (K_1^{-1})^{IJ} - \frac{2p_2}{2p_2\nu_1 + \alpha} \sum_{L,M}^{m+n} (K_1^{-1})^{IL} (K_1^{-1})^{JM} \\ \nu_2 &= \sum_{I,J}^{m+n} (K_2^{-1})^{IJ} = \frac{\nu_1}{2p_2\nu_1 + \alpha}. \end{aligned} \tag{5.4.34}$$

Comparison with equation (4.3.11) shows that ν_2 indeed corresponds to the second level of the composite fermion hierarchy.

We are now able to construct trial wave functions for stable states at fractions other than $\frac{1}{odd}$. Repeating the steps from (5.3.78) to (5.3.81), but now for vertex operators $V^I(x_i^I) = \,: e^{-iK_{IL}\varphi^L(x_i^I)} :$ that create a (composite) electron in the I'th Landau level at position x_i^I, and using the commutation relations (5.4.15), we get the following functions:

$$\psi_K = \prod_{M<N} \prod_{i,j} (z_i^M - z_j^N)^{K_{MN}} \prod_I \prod_{a,b} (z_a^I - z_b^I)^{K_{II}}. \tag{5.4.35}$$

They are antisymmetric in the electron positions iff K is integer with odd diagonal elements. Numerical studies have shown that these trial functions (with an extra non-analytic factor $\exp(-\frac{1}{2}\sum_{i,I}|z_i^I|^2)$ added) describe incompressible states.

5.5 Summary

The results of the effective action approach are the following: The pure Chern-Simons action, producing the macroscopic equations of motion for the electric current in the bulk in the absence of impurities, can be written as an action of massless chiral bosons on the edge of the sample. For every Landau level that conducts current, one boson appears in the edge action. The K-matrix, containing the current-current couplings in the Chern-Simons action, gives the couplings between the different bosons in the edge action.

The edge theory has a $\widehat{U(1)}^{\otimes m}$-symmetry, so the Hilbert space splits into representations of this Kač-Moody algebra. Every lowest weight representation corresponds to a separate kind of particle per Landau level, and the charges and statistics of these particles are contained in the K-matrix. We started with a matrix K of which we only knew that it is symmetric and invertible. However, by imposing that particles with unit charge obey anticommutation relations, K is constrained to be an integer matrix with odd diagonal elements. This leads to a particle spectrum containing composite fermions and particles with fractional charge and statistics.

Jain's hierarchy has a natural implementation in the effective action approach. The requirement that the electrons in every Landau level carry (approximately) the same number of flux quanta translates into requiring that the electromagnetic field interacts with all currents in (approximately) the same way. This gives a prescription for the K-matrix.

Chapter 6

Incompressibility and the $W_{1+\infty}$ algebra

> Infinite: Bigger than the biggest thing ever and then some. Much bigger than that in fact, really amazingly immense, a totally stunning size, real 'wow, that's big', time. Infinity is just so big that by comparison, bigness itself looks really titchy. Gigantic multiplied by colossal multiplied by staggeringly huge is the sort of concept we're trying to get across here.
>
> Douglas Adams, *The Restaurant at the End of the Universe*

In chapter 5 we wrote down an effective action for the Hall system at plateau values of the filling fraction and made the important observation that it actually describes one or more incompressible quantum fluids. We are now going to concentrate on this aspect of incompressibility. The area that is occupied by an incompressible fluid is conserved in time. Therefore, the Hamiltonian commutes with the group of area preserving diffeomorphisms. This is an infinite-dimensional group, generated by an infinite-dimensional algebra called the $W_{1+\infty}$ algebra.

As an illustration, we will first show that the algebra of area preserving diffeomorphisms is present in the theory of the quantum Hall effect from the very beginning; It can already be found both in the quantum and in the classical desription of noninteracting two-dimensional electrons in a magnetic field.

The meaning of the conformal invariance of the m-boson theory on the edge of the sample becomes clear. When acting on a droplet of incompressible fluid, the $W_{1+\infty}$ generators can only change the shape of the edge. These deformations induce conformal transformations of the edge coordinates $x^{\pm}$. Some generators create moving edge waves. These waves are identified as the edge currents found in chapter 5. Indeed, the generators that create them are shown to form a Kač-Moody algebra.

When a theory is invariant under the action of a certain group, the Hilbert space furnishes a representation of that group. The invariance of the quantum Hall Hamiltonian under area preserving diffeomorphisms indicates that the Hilbert space is a representation of the $W_{1+\infty}$ algebra. This fact provides us with a beautyful way of deriving very general and meaningful results *without knowing anything about the microscopic physics that underlie the QHE*. It is enough to know that the microscopic details of the hall system somehow manage to create an incompressible fluid, and then the representation theory of $W_{1+\infty}$ tells you what values of the Hall conductance σ_H are actually possible.

By explicit construction of the vector potential surrounding composite fermions, we then show that modifications of the $W_{1+\infty}$ generators, corresponding to the attachment of an even number of flux quanta to the electrons, do not change the commutation relations, thereby giving us new representations of $W_{1+\infty}$. The composite fermion trial wave functions ψ_K are found to be lowest weight states of these representations.

At the second quantised level it is seen that the algebra for m composite fermion fluids splits into m separate commuting $W_{1+\infty}$ algebras, each one associated to a linear combination of fluids, prescribed by the K-matrix in exactly the same way that the physical boson fields φ^I combine into the noninteracting field γ_a.

When the thermodynamic limit $N \to \infty$ is taken, the theory of the incompressible fluid becomes an edge theory. This confirms the idea that incompressible fluids are completely characterised by their edge dynamics. What is more, each one of the commuting algebras aquires a central charge $c = 1$ due to renormalisation effects.

These facts raise the hope, but certainly do not prove, that there is an intimate relation between $W_{1+\infty}$ representations and $\widehat{U(1)}^{\otimes m}$ representations.

To conclude we discuss the classification of $W_{1+\infty}$ representations by Kač et al. [23, 24]. They found that all unitary lowest weight representations are equivalent to unitary lowest weight representations of $\widehat{U(1)}^{\otimes m}$. The physical interpretation is that the mutually commuting $W_{1+\infty}$ edge algebras correspond to the decoupled fields in the edge model. In this way contact is made with the effective current theory.

One would like to have a way of predicting the stability of filling fractions from the point of view of $W_{1+\infty}$ representations. In the original construction of Jain's hierarchy it was argued that the stability of composite fermion states is related to the uniformity of the distribution of flux quanta over the electrons. In the effective edge model, this was expressed as uniformity of current-current coupling strengths. We discuss the proposal by Cappelli, Zemba and Trugenberger [27] that the stability of a state is related to the degeneracy of the $W_{1+\infty}$ weight vectors describing that state. They found that the first level of Jain's hierarchy, containing the most stable quantum Hall states, has the maximum possible degeneracy.

6.1 Infinite symmetry

In chapter 3 we presented the simple problem of a charged particle in two dimensions in the presence of a magnetic field. The Schrödinger equation was solved, yielding the one-particle wave functions ψ_{nj} expressed in terms of the laddering operators $a^\dagger$ and $b^\dagger$. The generalization to N noninteracting electrons was made. We are now going to use these simple ingredients again to illustrate the fact that even at a low level of sofistication all the aspects of incompressibility and $W_{1+\infty}$ representation theory are present in models of the quantum Hall effect.

6.1.1 The classical algebra w_∞

From equations (3.2.7) and (3.2.8) we see that the laddering operators b and $b^\dagger$ commute with a and $a^\dagger$, and consequently also with the hamiltonian $H \propto a^\dagger a$. Indeed, as Capelli, Zemba and Trugenberger (CZT) [25] pointed out, the whole enveloping algebra of b and $b^\dagger$ is therefore a symmetry of H. Using basic classical mechanics, they showed that this algebra generates area preserving diffeomorphisms.

Let us define the classical equivalents of a, b and H on the 4-dimensional phase space (x, y, π_x, π_y),

$$\begin{aligned} a_{cl} &:= \frac{m}{\ell eB}(v^2 - iv^1) = \frac{1}{\ell eB}[(\pi_y + \frac{Be}{2}x) - i(\pi_x - \frac{Be}{2}y)] \\ b_{cl} &:= -\frac{1}{\ell eB}[(\pi_y - \frac{Be}{2}x) + i(\pi_x + \frac{Be}{2}y)] \\ H_{cl} &:= \frac{1}{2}m\vec{v}^2 = \hbar\omega_c a_{cl}^* a_{cl}. \end{aligned} \tag{6.1.1}$$

The Poisson bracket is defined as

$$\{A, B\} := \frac{\partial A}{\partial x^i}\frac{\partial B}{\partial \pi_i} - \frac{\partial B}{\partial x^i}\frac{\partial A}{\partial \pi_i}. \tag{6.1.2}$$

With these definitions we get the classical equivalents of the commutation relations,

$$\begin{aligned} \{a_{cl}, a_{cl}^*\} &= -\frac{i}{\hbar} \\ \{b_{cl}, b_{cl}^*\} &= -\frac{i}{\hbar} \\ \{a_{cl}^{(*)}, b_{cl}^{(*)}\} &= 0. \end{aligned} \tag{6.1.3}$$

The third equation shows that the enveloping algebra of b_{cl} and b_{cl}^* commutes with the classical Hamiltonian. A convenient basis for this algebra is given by

$$\begin{aligned} w_{nm} &:= (b_{cl}^*)^n (b_{cl})^m, \quad n, m \geq 0, \\ \{w_{nm}, H_{cl}\} &= 0. \end{aligned} \tag{6.1.4}$$

The w_{nm} span an algebra known as w_∞, the classical algebra of area preserving diffeomorphisms,

$$\{w_{nm}, w_{kl}\} = -\frac{i}{\hbar}(mk - nl)w_{n+k-1,m+l-1}. \tag{6.1.5}$$

The presence of this algebra in this particular form can be understood as follows: The w_{nm} generate canonical transformations in the standard way,

$$\delta x^i = \frac{\partial w_{nm}}{\partial \pi_i} \quad ; \quad \delta \pi_i = -\frac{\partial w_{nm}}{\partial x^i}. \tag{6.1.6}$$

These canonical transformations leave the Hamiltonian invariant and, what is more, the velocities v_x and v_y are also invariant, since they are proportional to $\mathrm{Im}(a_{cl})$ and $\mathrm{Re}(a_{cl})$ respectively. This means that the w_{nm} do not act on the complete four-dimensional phase space, but only on a two-dimensional subspace $\mathcal{E}(\vec{v})$ characterized by constant velocity $\vec{v}$ (or a_{cl}). The two remaining degrees of freedom are contained in b_{cl},

$$b_{cl} = \frac{1}{\ell}(x - iy) = \bar{z}. \tag{6.1.7}$$

This subspace is exactly the coordinate space (x, y) of the electron. By Liouville's theorem, canonical transformations leave the volume element of phase space invariant. Therefore, the transformations generated by w_{nm} do not change the two-dimensional volume element $dx \wedge dy$, i.e. they are area preserving diffeomorphisms.

In order to gain a better understanding of these transformations, we can choose an arbitrary subspace $\mathcal{E}(\vec{v})$ to represent the whole phase space and study the effects of w_∞ on this representative subspace. For convenience we take $\vec{v}=0$. If we view the complex coordinates z and $\bar{z}$ as independent, the Poisson bracket in $\mathcal{E}(0)$ takes a very nice form,

$$\{A,B\} = \frac{4i}{\hbar}(\bar{\partial}A\partial B - \bar{\partial}B\partial A). \tag{6.1.8}$$

We see that z and $\bar{z}$ are canonically conjugate.

The w_∞ generators are also drastically simplified,

$$w_{nm} = z^n \bar{z}^m. \tag{6.1.9}$$

Let us now take a circular droplet of incompressible quantum fluid with constant density ρ_0 and radius R. The density distribution ρ is given by

$$\rho(z,\bar{z}) = \rho_0 \Theta(R^2 - z\bar{z}), \tag{6.1.10}$$

where Θ is the Heaviside step function. Taking the Poisson bracket (6.1.8) of ρ with a generator w_{nm}, we find that the effect of an area preserving diffeomorphism on the incompressible droplet is the creation of an edge wave,

$$\triangle_{n,m}\rho(z,\bar{z}) = \{\rho, w_{nm}\} = -\frac{4i}{\hbar}\rho_0 R^{n+m}(n-m)e^{i(n-m)\theta}\delta(R^2 - z\bar{z}), \tag{6.1.11}$$

where θ is the azimuthal coordinate. The momentum of the wave is $n-m$.

6.1.2 The quantum algebra $W_{1+\infty}$

It is convenient to choose a basis of the enveloping algebra of the first quantised operators b, $b^\dagger$ in analogy with (6.1.4),

$$W_{nm} := (b^\dagger)^n b^m, \qquad n,m \geq 0. \tag{6.1.12}$$

These generators are not hermitian; we have

$$W^\dagger_{nm} = W_{mn}. \tag{6.1.13}$$

Unitary transformations are generated by the Hermitian combinations $W_{nm}+W_{mn}$ and $i(W_{nm}-W_{mn})$. The commutation relations of the basis elements are found using (3.2.8),

$$[W_{nm}, W_{kl}] = \sum_{s=1}^{\min(m,k)} s! \begin{pmatrix} m \\ s \end{pmatrix} \begin{pmatrix} k \\ s \end{pmatrix} W_{n+k-s,m+l-s} - \begin{bmatrix} m \leftrightarrow l \\ n \leftrightarrow k \end{bmatrix}. \tag{6.1.14}$$

The leading ($s=1$) term exactly reproduces the w_∞ commutation rules. The other terms are 'quantum corrections'. Quantum extensions of w_∞ are collectively called W_∞. The particular algebra (6.1.14) is known by the curious name of $W_{1+\infty}$, where the '1' refers to the fact that the algebra contains generators with conformal weight 1.

$W_{1+\infty}$ has some interesting subalgebras, which are also present in w_∞:

$$\begin{aligned} [W_{1,n+1}, W_{1,m+1}] &= (n-m)W_{1,n+m+1} \\ [W_{n+1,1}, W_{m+1,1}] &= (m-n)W_{n+m+1,1} \\ [W_{nn}, W_{mm}] &= 0 \qquad (\mathit{Cartan\ subalgebra}). \end{aligned} \tag{6.1.15}$$

An important element of the Cartan subalgebra is $W_{11} = b^\dagger b$. It measures the angular momentum of a state. If the degeneracy of the Hamiltonian with respect to the angular momentum is lifted by the application of a radial electric field (see (3.4.4)), then W_{11} also measures the extra energy due to the electric field.

The first two relations of (6.1.15) form the two halves of an algebra that looks like the classical Virasoro algebra, with the identification

$$V_n \approx W_{1,n+1} \quad V_{-n} \approx W_{n+1,1} \quad n \geq 0. \tag{6.1.16}$$

However, the two halves do not form a whole Virasoro algebra; In the Virasoro algebra we would have $[V_n, V_{-m}] = (n+m)V_{n-m}$, but $[W_{n+1,1}, W_{1,m+1}]$ does not give $W_{n-m+1,1}$ or $W_{1,n-m+1}$.

The Virasoro algebra arises as a subalgebra of w_∞ in the following way: Since z and $\bar{z}$ are canonically conjugate, we can construct a Bargmann representation in which $\bar{z}$ is written as $\frac{\partial}{\partial z}$, giving $w_{n+1,1}$ the form of the classical Virasoro generator $z^{n+1}\partial_z$ (see (5.3.6)), or a representation where z is written as $\frac{\partial}{\partial \bar{z}}$, giving $w_{1,n+1}$ the form $\bar{z}^{n+1}\partial_{\bar{z}}$.

From these forms we see that the $w_{n+1,1}$ and $w_{1,n+1}$ generate conformal reparametrisations of the complex plane as follows

$$\begin{array}{lcl} w_{n+1,1} + w_{1,n+1} & : & \left\{ \begin{array}{lcl} z & \to & z + \epsilon z^{n+1} \\ \bar{z} & \to & \bar{z} + \epsilon \bar{z}^{n+1} \end{array} \right. \\ i(w_{n+1,1} - w_{1,n+1}) & : & \left\{ \begin{array}{lcl} z & \to & z + i\epsilon z^{n+1} \\ \bar{z} & \to & \bar{z} - i\epsilon \bar{z}^{n+1} \end{array} \right. , \quad n \geq -1. \end{array} \tag{6.1.17}$$

The negative modes are missing, since for $n < -1$ the reparametrisations would be singular in the origin $z = 0$.

6.2 Composite fermion representations

Following Flohr and Varnhagen [19] and Karabali [28] we present lowest weight representations of $W_{1+\infty}$ that describe a collection of incompressible fluids, made of composite fermions. These representations exactly correspond to the physical picture given in chapter 5 for the effective edge theory.

6.2.1 Trial wave functions as $W_{1+\infty}$ lowest weight states

For a system of N noninteracting electrons, the W_{nm} (6.1.12) are generalised to differential operators acting on N-particle wave functions,

$$W_{nm} = \sum_{i=1}^{N} (b_i^\dagger)^n b_i^m, \tag{6.2.1}$$

where the index i refers to the i'th electron. These operators commute with the N-particle hamiltonian. They also obey the algebra (6.1.14). W_{11} measures the total angular momentum of an N-electron wave function, and W_{00} measures its total charge.

Let us now bind $-2p$ flux quanta to all electrons ($p \in \mathbb{N}$), so that they become composite fermions with statistics $\frac{\theta}{\pi} = 2p + 1$. The fluxes give a new contribution to the vector potential, circling around the electrons. The covariant derivatives are affected by this extra contribution, and therefore also the differential operators a_i, $a_i^\dagger$, b_i and $b_i^\dagger$ at the position of the i'th particle. It is possible to choose a gauge in which these operators have a particularly nice form:

$$\begin{aligned} a_i^\dagger \to \tilde{a}_i^\dagger &:= \Theta^{2p} \circ a_i^\dagger \circ \Theta^{-2p} = a_i^\dagger + 2p \sum_{j|j\neq i}^{N} \frac{1}{z_i - z_j} \\ b_i \to \tilde{b}_i &:= \Theta^{2p} \circ b_i \circ \Theta^{-2p} = b_i - 2p \sum_{j|j\neq i}^{N} \frac{1}{z_i - z_j} \\ a_i \to \tilde{a}_i &:= \Theta^{2p} \circ a_i \circ \Theta^{-2p} = a_i \\ b_i^\dagger \to \tilde{b}_i^\dagger &:= \Theta^{2p} \circ b_i^\dagger \circ \Theta^{-2p} = b_i^\dagger, \end{aligned} \tag{6.2.2}$$

with Θ defined as the following function of the N particle positions,

$$\Theta(x_1, \cdots, x_N) := \prod_{i<j}^{N} (z_i - z_j). \tag{6.2.3}$$

The commutation relations are also transformed. Using the relation $\partial \bar{z}^{-1} = \bar{\partial} z^{-1} = \pi\delta(z)$ we get the following nonzero commutators for $i \neq j$,

$$[\tilde{b}_i, \tilde{b}_j^\dagger] = [\tilde{a}_i, \tilde{b}_j] = -[\tilde{a}_i, \tilde{a}_j^\dagger] = -[\tilde{a}_i^\dagger, \tilde{b}_j^\dagger] = 2p\pi\delta(z_i - z_j). \tag{6.2.4}$$

For $i = j$ we get

$$\begin{aligned} [\tilde{a}_i, \tilde{a}_i^\dagger] &= 1 + 2p\pi \sum_{k|k\neq i} \delta(z_i - z_k) \\ [\tilde{b}_i, \tilde{b}_i^\dagger] &= 1 - 2p\pi \sum_{k|k\neq i} \delta(z_i - z_k) \\ [\tilde{a}_i^\dagger, \tilde{b}_i^\dagger] = -[\tilde{a}_i, \tilde{b}_i] &= 2p\pi \sum_{k|k\neq i} \delta(z_i - z_k), \end{aligned} \tag{6.2.5}$$

with all other commutators zero. Note that the commutators of the transformed operators are not equal to the transformed commutators of the original operators, due to the fact that the transformation is singular.

The transformed N-particle Hamiltonian is given by

$$\tilde{H} = \frac{1}{2}\hbar\omega_c \sum_{i=1}^{N} (\tilde{a}_i^\dagger \tilde{a}_i + \tilde{a}_i \tilde{a}_i^\dagger) = \hbar\omega_c \left[\sum_{i=1}^{N} (\tilde{a}_i^\dagger \tilde{a}_i + \frac{1}{2}) + 2p\pi \sum_{i<j}^{N} \delta(z_i - z_j) \right]. \tag{6.2.6}$$

In addition to the old kinetic term there is now a repulsive δ-interaction term between all electron pairs. The enveloping algebra of $\tilde{b}_i$, $\tilde{b}_i^\dagger$ is a symmetry algebra of the transformed Hamiltonian. A basis can be chosen of the form (6.2.1),

$$\begin{aligned} W_{nm}^p &:= \sum_{i=1}^{N} (\tilde{b}_i^\dagger)^n (\tilde{b}_i)^m \\ [W_{nm}^p, \tilde{H}] &= 0. \end{aligned} \tag{6.2.7}$$

The commutation relations of the W^p_{nm} are given by the $W_{1+\infty}$ algebra, apart from extra terms containing δ-functions and derivarives thereof,

$$\begin{aligned}[W^p_{nm}, W^p_{kl}] &= \sum_{s=1}^{\min(m,k)} s! \begin{pmatrix} m \\ s \end{pmatrix} \begin{pmatrix} k \\ s \end{pmatrix} W^p_{n+k-s,m+l-s} - \begin{bmatrix} m \leftrightarrow l \\ n \leftrightarrow k \end{bmatrix} \\ &+ p \sum_{i \neq j} \begin{matrix} \textit{terms involving} \\ \delta(z_i - z_j) \end{matrix} . \end{aligned} \tag{6.2.8}$$

The probability of finding two (composite) fermions at the same position is zero, because of the Pauli principle. Therefore, all δ-functions disappear from the commutation relations of the transformed operators when the function space these operators act on is restricted to (composite) fermionic wave functions. The algebra of the symmetry generators W^p_{nm}, restricted in this way, is precisely $W_{1+\infty}$.

Flohr and Varnhagen [19] showed that the Laughlin wave functions $\psi_p(z_1, \cdots, z_N)$ (4.1.7) are lowest weight states of the $W_{1+\infty}$-algebra:

$$\begin{aligned} W^p_{nm}\psi_p &= 0, \quad n - m < 0 \\ W^p_{nn}\psi_p &= n! \begin{pmatrix} N \\ n+1 \end{pmatrix} \psi_p. \end{aligned} \tag{6.2.9}$$

A complete representation is built by acting on the ground state with all W^p_{nm} $(n-m > 0)$. In this way chiral edge waves are created, with angular momentum W^p_{11} given by

$$W^p_{11} W^p_{nm}\psi_p = (n-m) W^p_{nm}\psi_p, \quad n - m > 0 \tag{6.2.10}$$

Note that the W^p_{nn}-eigenvalues do not depend on p. The Laughlin ground states have the form $\psi_p = \Theta^{2p}\psi_{p=0}$, from which it follows that the transformed operators (6.2.2) act on ψ_p in the same way as the untransformed operators on ψ_0.

In fact, it can be shown that all expectation values (not only in the ground states) are independent of p, in perfect agreement with the idea that attaching an even number of flux tubes to all electrons does not change the physics: Due to the particular gauge that we chose, the transformation (6.2.2) contains an asymmetry; the $a_i^\dagger$ and the b_i aquire extra terms, while the a_i and $b_i^\dagger$ do not. In order to ensure that $\tilde{a}_i, \tilde{b}_i$ and $\tilde{a}_i^\dagger, \tilde{b}_i^\dagger$ are each other's complex conjugate, a new integration measure μ_p is needed,

$$\mu_p(x_1, \cdots, x_N) := |\Theta|^{-4p}. \tag{6.2.11}$$

The inner product of two composite fermion wave functions f_p and g_p is given by

$$(f_p, g_p) = \int d^2z_1 \cdots d^2z_N \; f_p^* \mu_p g_p. \tag{6.2.12}$$

Composite fermion wave functions, constructed from the ground state ψ_p, are of the form $f_p = \Theta^{2p} f_0$. This gives

$$(f_p, g_p) = (f_0, g_0), \tag{6.2.13}$$

i.e. the inner product is invariant under the transition from electrons to composite fermions.

The above picture for $\nu = \frac{1}{2p+1}$ is easily generalised to a system of m quantum fluids with $\nu = \sum_{I,J}(K^{-1})^{IJ}$, where K is the $m \times m$ matrix containing the coupling strengths between the fluids as in (5.4.4). The positions of the electrons get an extra label, corresponding to the fluid they belong to. The transformation (6.2.2) is now written as

$$\begin{aligned} a_i^{I\dagger} \to \tilde{a}_i^{I\dagger} &:= a_i^{I\dagger} + \sum_{J|J\neq I} K_{IJ} \sum_j \frac{1}{z_i^I - z_j^J} + (K_{II}-1) \sum_{j|j\neq i} \frac{1}{z_i^I - z_j^I} \\ b_i^I \to \tilde{b}_i^I &:= b_i^I - \sum_{J|J\neq I} K_{IJ} \sum_j \frac{1}{z_i^I - z_j^J} - (K_{II}-1) \sum_{j|j\neq i} \frac{1}{z_i^I - z_j^I} \end{aligned} \tag{6.2.14}$$

and the symmetry generators have the form

$$W_{nm}^p = \sum_{I,i} (\tilde{b}_i^{I\dagger})^n (\tilde{b}_i^I)^m. \tag{6.2.15}$$

The lowest weight states have the form of the trial wave functions (5.4.35), obtained as N-point correlators in the effective edge theory,

$$\psi_K = \exp(-\frac{1}{2}\sum_{I,i} |z_i^I|^2) \prod_{M<N} \prod_{i,j} (z_i^M - z_j^N)^{K_{MN}} \prod_I \prod_{a<b} (z_a^I - z_b^I)^{K_{II}}. \tag{6.2.16}$$

The fact that the lowest weight states of $\widehat{U(1)}^{\otimes m}$ and $W_{1+\infty}$ describe the same composite fermions, with interactions given by the same K-matrix, indicates that there is a strong connection between the representations of $\widehat{U(1)}^{\otimes m}$ and $W_{1+\infty}$.

6.3 $W_{1+\infty}$ on the edge

Extra insight in the relation between the effective edge theory and the $W_{1+\infty}$ representations can be gained by writing the $W_{1+\infty}$ algebra in a second quantised form and then taking the thermodynamic limit. When written in second quantised form, it is seen that the algebra splits into mutually commuting copies of $W_{1+\infty}$, one for every quantum fluid that is described. The theory of an infinitely big incompressible droplet becomes an edge theory. What is more, it turns out that due to the renormalisation necessary in the limit $N \to \infty$, every copy of the $W_{1+\infty}$ algebra aquires a central charge $c = 1$, yielding the same total central charge as in the Kač-Moody algebra of the effective edge theory.

6.3.1 Field quantisation

Our system of noninteracting electrons is easily described in this formalism. A scalar electron field Ψ is introduced, satisfying the Schrödinger equation

$$i\hbar D_0 \Psi = \frac{\hbar^2}{2m} D_i D^i \Psi, \tag{6.3.1}$$

where D_μ is the gauge covariant derivative for a constant magnetic field perpendicular to the plane. The field is quantized using as a basis the single particle wave functions ψ_{nj} defined in (3.3.3),

$$\Psi(z,\bar{z}) = \sum_{n=0}^{\infty} \sum_{j=-n}^{\infty} f_{nj} \psi_{nj}(z,\bar{z}), \tag{6.3.2}$$

$$f_{rk} = \int d^2z \; \psi_{rk}^* \Psi. \tag{6.3.3}$$

The f_{nj} are fermionic operators on Fock space that annihilate the states ψ_{nj}. They obey the standard anticommutation relations

$$\{f_{nj}, f^\dagger_{n'j'}\} = \delta_{nn'}\delta_{jj'} \tag{6.3.4}$$

with all other anticommutators zero. The vacuum is defined by the relation

$$f_{nj}|0>= 0 \qquad \forall\, n, j. \tag{6.3.5}$$

The field quantised analogues of the generators W_{nm} are given by

$$\mathcal{W}_{nm} := \int d^2z\ \Psi^\dagger (b^\dagger)^n (b)^m \Psi, \quad n, m \geq 0. \tag{6.3.6}$$

They all commute with the Hamiltonian $H = \frac{\hbar^2}{2m}\int d^2x\ (D_i\Psi)^\dagger(D_i\Psi)$. Using the anticommutation relations (6.3.4), it is seen that the $\mathcal{W}_{nm}$ too obey the $W_{1+\infty}$ commutation relations (6.1.14).

By substitution of (6.3.2) into the definition (6.3.6) of the $\mathcal{W}_{nm}$ we find that they have the form of a sum of parts that act only within a single Landau level;

$$\mathcal{W}_{nm} = \sum_{r=0}^{\infty} \mathcal{W}^{(r)}_{nm}, \tag{6.3.7}$$

where r labels the Landau level and $\mathcal{W}^{(r)}_{nm}$ is given by

$$\mathcal{W}^{(r)}_{nm} := \sum_{k=m}^{\infty} \frac{\sqrt{k!(k+n-m)!}}{(k-m)!} f^\dagger_{r,k+n-m} f_{r,k}. \tag{6.3.8}$$

What is more, the $\mathcal{W}^{(r)}_{nm}$ span mutually commuting closed algebras, one for each Landau level, which separately obey the $W_{1+\infty}$ commutation relations (6.1.14). These Landau level subalgebras each contain the Cartan and classical Virasoro subalgebras (6.1.15).

Expression (6.3.8) tells us something about the action of the $\mathcal{W}^{(r)}_{nm}$ on Hilbert space. The bilinear $f^\dagger_{r,k+n-m} f_{r,k}$ displaces an electron inward or outward by changing its angular momentum by an amount $n - m$. A negative (positive) change corresponds to a compression (decompression). Since the electrons are only displaced, the particle number is invariant under $W_{1+\infty}$. Equation (6.3.8) can be inverted as follows,

$$f^\dagger_{ra} f_{rb} = \frac{1}{\sqrt{a!b!}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mathcal{W}^{(r)}_{a+k,b+k}, \tag{6.3.9}$$

showing that the sets $\{\mathcal{W}^{(r)}_{nm}\}^{\infty}_{n,m=0}$ and $\{f^\dagger_{ra} f_{rb}\}^{\infty}_{a,b=-r}$ are equivalent. Every thinkable displacement can be accomplished by acting on a state with a suitable element of $W_{1+\infty}$. Therefore, the Hilbert space $\mathcal{H}^{(r)}_N$ of N electrons in the r'th Landau level is completely spanned by $\{\mathcal{W}^{(r)}_{nm}\}$ acting on an arbitrary N-electron state; The Hilbert space $\mathcal{H}^{(r)}_N$ is a representation of $W_{1+\infty}$.

A lowest weight representation is constructed by taking a filled r'th Landau level $|\Omega_r>$ as the lowest weight state. This state is created out of the vacuum by placing N electrons in the N lowest angular momentum states,

$$|\Omega_r > := \prod_{j=-r}^{N-r-1} f^\dagger_{rj}|0> . \tag{6.3.10}$$

Using (6.3.3), we can write this state as

$$\begin{aligned} |\Omega_r> &= \int d^2z_1 \cdots d^2z_N \left[\prod_{k=1}^{N} \psi_{r,k-r-1}(z_k)\Psi^\dagger(z_k)\right] |0> \\ &=: \int d^2z_1 \cdots d^2z_N \left[\prod_{k=1}^{N} \psi_{r,k-r-1}(z_k)\right] |z_1 \cdots z_N> . \end{aligned} \tag{6.3.11}$$

Due to the total antisymmetry of $|z_1 \cdots z_N>$ in all pairs of electron positions, we can also write

$$|\Omega_r> = \frac{1}{N!} \int d^2z_1 \cdots d^2z_N \left[\mathcal{A} \prod_{k=1}^{N} \psi_{r,k-r-1}(z_k)\right] |z_1 \cdots z_N>, \tag{6.3.12}$$

where $\mathcal{A}$ stands for total antisymmetrisation. The factor between square brackets is exactly the Slater determinant for N noninteracting particles. For $r = 1$ we obtain the $\nu = 1$ Laughlin wave function.

The electron density $\rho = \Psi^\dagger\Psi$ in the state $|\Omega_r>$ is given by

$$< \Omega_r|\rho(z,\bar{z})|\Omega_r> = \frac{1}{\pi} e^{-|z|^2} \sum_{j=0}^{N-r-1} \frac{|z|^{2j}}{j!} [L_r^j(|z|^2)]^2. \tag{6.3.13}$$

For large N, this describes a droplet of radius $\sqrt{N}$; The density is a constant $1/\pi$ inside this radius and rapidly drops to zero at $|z| \approx \sqrt{N}$. The electron density gradient at the edge of the droplet gives rise to an electric field in the radial direction, leading to an electric current in the azimuthal direction,

$$\begin{aligned} j_k &:= -e\frac{\hbar}{2mi}\left(\Psi^\dagger D_k \Psi - h.c.\right) \\ < \Omega_r|j^i|\Omega_r> &= -\frac{\hbar}{2m}\varepsilon^{ij}\partial_j < \Omega_r|\rho|\Omega_r> . \end{aligned} \tag{6.3.14}$$

There is no current in the interior of the droplet.

The lowest weight condition reads

$$\begin{aligned} \mathcal{W}_{nm}^{(r)}|\Omega_r> &= 0, \qquad n-m<0. \\ \mathcal{W}_{nn}^{(r)}|\Omega_r> &= \sum_{k=n}^{N-r-1} \frac{k!}{(k-n)!}|\Omega_r> . \end{aligned} \tag{6.3.15}$$

The first relation reflects the incompressibility of the filled LL; a compression annihilates the ground state. The second relation tells that $|\Omega_r>$ is an eigenstate of the Cartan subalgebra.

The transformations of the droplet generated by the $\mathcal{W}_{nm}^{(r)}$ $(n-m>0)$ are similar to the canonical transformations generated by the w_{nm} in the classical picture. Electrons at the edge of the droplet are shifted outward by $n-m$ positions. Using the commutation relation (6.1.14) we find

$$[\mathcal{W}_{11}^{(r)}, \mathcal{W}_{nm}^{(r)}] = (n-m)\mathcal{W}_{nm}^{(r)}, \tag{6.3.16}$$

which shows that an excitation generated by $\mathcal{W}_{nm}^{(r)}$ has positive angular momentum $n-m$.

The transition (6.2.14) from electrons to composite fermions can also be made in second quantisation. For every quantum fluid we define one electron field operator Ψ^I, *restricted* *to the lowest Landau level*; in general, and especially in the case of Jain's hierarchy, the K-matrix describes states with filling fraction $\nu < 1$, so the physical electrons are in the lowest Landau level, even though the composite fermions occupy higher Landau levels. The field operators Ψ^I are mutually anticommuting,

$$\begin{aligned} \{f^I_{0j}, f^{J\dagger}_{0k}\} &= \delta^{IJ}\delta_{jk} \\ \{f^I_{0j}, f^J_{0k}\} &= \{f^{I\dagger}_{0j}, f^{J\dagger}_{0k}\} = 0. \end{aligned} \tag{6.3.17}$$

The transformation of the vector potential $\vec{A}$ that attaches flux quanta to the electrons is accomplished by adding a Chern-Simons interaction term to the free electron action (see (5.1.11)), with coupling σ_H determined by the K-matrix. It is again possible to choose a gauge in which $b^\dagger$ and a are unaltered and b and $a^\dagger$ have a simple form,

$$\begin{aligned} a^{I\dagger} \to \tilde{a}^{I\dagger} &:= a^{I\dagger} + \sum_J \hat{K}_{IJ} \int d^2z' \, \frac{\rho^J(z')}{z - z'} \\ b^I \to \tilde{b}^I &:= b^I - \sum_J \hat{K}_{IJ} \int d^2z' \, \frac{\rho^J(z')}{z - z'}. \end{aligned} \tag{6.3.18}$$

with $\hat{K}_{IJ} := K_{IJ} - \delta_{IJ}$ and $\rho^J := \Psi^{J\dagger}\Psi^J$. For every fluid we get generators $\mathcal{W}^{K(I)}_{nm}$, defined as

$$\mathcal{W}^{K(I)}_{nm} := \int d^2z \; \Psi^{I\dagger}(z)(\tilde{b}^\dagger)^n \tilde{b}^m \Psi^I(z). \tag{6.3.19}$$

The vector potential can be transformed away with a singular transformation like (5.1.20), changing the fermionic operator Ψ^I to a composite fermionic operator $\tilde{\Psi}^I$,

$$\begin{aligned} \tilde{\Psi}^I(z) &:= e^{-\hat{K}_{IJ}\Lambda^J(z)}\Psi^I(z), \\ \Lambda^J(z) &:= \int d^2z' \; \ln(z - z')\rho^J(z'). \end{aligned} \tag{6.3.20}$$

The generators $\mathcal{W}^{K(I)}_{nm}$ can now be written as

$$\mathcal{W}^{K(I)}_{nm} = \int d^2z \; \tilde{\Psi}^{I\dagger}(z)(b^\dagger)^n(b)^m \tilde{\Psi}^I(z). \tag{6.3.21}$$

Karabali [28] showed that these operators give rise to m mutually commuting $W_{1+\infty}$ algebras. The $\mathcal{W}$ generators have the following commutation relations:

$$[\mathcal{W}^{K(I)}_{nm}, \mathcal{W}^{K(J)}_{kl}] = \delta^{IJ}\left[\sum_{s=1}^{\min(m,k)} s! \begin{pmatrix} m \\ s \end{pmatrix}\begin{pmatrix} k \\ s \end{pmatrix} \mathcal{W}^{K(I)}_{n+k-s,m+l-s} - \begin{pmatrix} m \leftrightarrow l \\ n \leftrightarrow k \end{pmatrix}\right]. \tag{6.3.22}$$

These commuting algebras, constructed from the anticommuting fields $\tilde{\Psi}^I$, correspond to the commuting Kač-Moody algebras of the noninteracting fields γ_a in the edge boson model. The W-algebras that correspond to the physical fields φ^I are given by the linear combinations $\sum_J (K^{-1})^{IJ}\mathcal{W}^{K(J)}_{nm}$. This is exactly to what happens to the Kač-Moody algebra $\widehat{U(1)}^{\otimes m}$ in the edge model. The symmetry algebra decouples into m independent parts that are associated not to the physical fields, but to linear combinations thereof, given by the K-matrix.

6.3.2 Renormalisation

The fact that something is infinite does not necessarily mean it is zero.

S. Weinberg

From the second equation in (6.3.15) we see that all eigenvalues of the Cartan subalgebra $\mathcal{W}_{nn}^{(r)}$ go to infinity in the thermodynamic limit. These infinities can be cancelled by performing a normal ordering with respect to the lowest weight state $|\Omega_r>$,

$$:A:=A-<\Omega_r|A|\Omega_r>, \tag{6.3.23}$$

which corresponds to a subtraction of the filled Landau level energy. The lowest weight state $|\Omega_r>$ becomes the new vacuum; The filled LL plays the role of infinite Dirac sea; The particles created from the new vacuum are edge waves.

CZT [26] showed that after renormalisation with respect to the filled lowest Landau level $|\Omega_0>$, the generators $\mathcal{W}_{nm}^{(r)}$ in the *lowest* Landau level $r=0$ have the form (6.3.6) but now with $b, b^\dagger$ restricted to constant radius and with Ψ proportional to the field operator of a relativistic chiral edge fermion ψ. This can be understood as follows: The hamiltonian of our system of electrons in a magnetic field is given by

$$H=\frac{\hbar^2}{2m}\int d^2x\ (D_i\Psi)^\dagger(D_i\Psi). \tag{6.3.24}$$

The integrand can be rewritten as

$$\begin{aligned}(D_i\Psi)^\dagger(D_i\Psi) &= ([D_1+iD_2]\Psi)^\dagger[D_1+iD_2]\Psi \\ &- \frac{i}{2}\varepsilon^{ij}\partial_i[\Psi^\dagger D_j\Psi-(D_j\Psi)^\dagger\Psi]+2\ell^{-2}\Psi^\dagger\Psi.\end{aligned} \tag{6.3.25}$$

If we restrict Ψ to the lowest Landau level, then we have $[D_1+iD_2]\Psi=0$, since the single-electron wave functions ψ_{0j} that this Ψ is composed of satisfy $D_{\bar{z}}\psi_{0j}=0$. The last term of (6.3.25) corresponds to a constant shift of the single-electron energy and can therefore be ignored. We are left with a hamiltonian on the edge,

$$H=\frac{-i}{8}\oint d\theta\ (\Psi^\dagger D_\theta\psi-h.c.), \tag{6.3.26}$$

where θ is the azimuthal coordinate and H and Ψ have been made dimensionless. Equation (6.3.26) has precisely the form of the hamiltonian for a right-handed relativistic fermion in $1+1$ dimensions. This calculation actually constitutes an entirely independent way of showing that the dynamics of the Quantum Hall system can be formulated solely in terms of observables on the edge.

The single-electron wave functions ψ_{0j}, when restricted to constant radius, have a very simple dependence on the azimuthal coordinate: $\psi_{0j}\propto e^{ij\theta}$. Therefore it is possible to describe the electrons on the edge with a chiral field ψ that is expanded in the standard way,

$$\psi(\theta)=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}a_k e^{i(k-\frac{1}{2})\theta}. \tag{6.3.27}$$

The operators a_k are given by the f_{0k}, redefined with respect to the new vacuum,

$$\begin{aligned}a_j &:= f_{0,j+N} \\ a_j|\Omega_0>= 0 &= a^\dagger_{-j-1}|\Omega_0>, \quad j\geq 0.\end{aligned} \tag{6.3.28}$$

Now a new basis for $W_{1+\infty}$ can be introduced,

$$
\begin{aligned}
V_n^j &= \int d\theta \; :[e^{-i\frac{\theta}{2}}\psi^\dagger(\theta)]e^{-in\theta}g_n^j(i\partial_\theta)[e^{i\frac{\theta}{2}}\psi(\theta)]: \\
&= \sum_{k=-\infty}^{\infty} g_n^j(-k) : a_{k-n}^\dagger a_k : \qquad (6.3.29)
\end{aligned}
$$

where g_n^j ($n \in \mathbb{Z}$, $j \geq 0$) is a polynomial of order j, defining the basis. It is chosen in such a way that $(V_n^j)^\dagger = V_{-n}^j$. The commutators of the basis elements are given by

$$
[V_n^i, V_m^j] = (jn - im)V_{n+m}^{i+j-1} + q(i,j,n,m)V_{n+m}^{i+j-3} + \cdots + \delta_{n+m,0}c(i,j,n) \qquad (6.3.30)
$$

where q and c are polynomials that depend on the specific choice for g_n^j. The polynomial c is a central extension, appearing here as a renormalisation effect. CZT chose to require that $c(i,j,n) \propto \delta^{ij}$. This uniquely defines a basis. The g_n^j in this basis become increasingly complicated with growing j. For $j = 0, 1$ we have relatively simple expressions

$$
\begin{aligned}
V_n^0 &= \sum_{j=-\infty}^{\infty} : a_{j-n}^\dagger a_j : \\
V_n^1 &= \sum_{j=-\infty}^{\infty} (j - \frac{n+1}{2}) : a_{j-n}^\dagger a_j : \qquad (6.3.31)
\end{aligned}
$$

We recognize these as current modes and Virasoro generators respectively (see 5.3.38), with $c = 1$ algebras,

$$
\begin{aligned}
[V_n^0, V_m^0] &= cm\delta_{n+m,0} \\
[V_n^1, V_m^1] &= (n-m)V_{n+m}^1 + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\
[V_n^1, V_m^0] &= -mV_{n+m}^0. \qquad (6.3.32)
\end{aligned}
$$

We now have a *complete* Virasoro subalgebra, not merely the positive modes. After taking the thermodynamic limit, all physical quantities are located on the edge, so the origin of the (x,y)-plane does not play a role any more and negative modes of z are allowed.

A lowest weight state is an eigenstate of the Cartan subalgebra V_0^j that is annihilated by all V_n^j with $n > 0$. A complete lowest weight representation is constructed by acting on the lowest weight state with all V_n^j, $n < 0$.

In a model with m incompressible fluids, described by lowest Landau level electron fields $\Psi^1, \cdots, \Psi^m$ (6.3.17), the decoupled algebras $\mathcal{W}_{nm}^{K(I)}$ each take the form (6.3.30). They are mutually commuting and the total $W_{1+\infty}$ algebra for the m-fluid system has central charge $c = m$.

6.4 Some representation theory of $W_{1+\infty}$

The aspect of incompressibility is present in every description of the quantum Hall effect: When using electron wave functions and field operators, one finds that $W_{1+\infty}$ is a symmetry algebra of the hamiltonian and that the ground states are lowest weight states of this algebra; In the effective current theory it turns out that the pure Chern-Simons action for the currents describes an incompressible system.

One is led to try the opposite approach: Let's assume that incompressibility is *the* fundamental, necessary condition for the existence of the quantum Hall effect. Then the purely mathematical theory of $W_{1+\infty}$ representations should provide a means of finding all possible values of the Hall conductance. As we are dealing with quantum theory, we are only interested in unitary lowest weight representations.

6.4.1 Unitary, irreducible lowest weight representations

The general $W_{1+\infty}$ representation theory was developed by Kač and Radul [23] and Frenkel, Kač, Radul and Wang [24] in 1993–1994.

They found that unitary, irreducible lowest weight representations of $W_{1+\infty}$ exist *only if the central charge is a positive integer*, $c = m$. The lowest weight states are characterised by an m-dimensional real-valued weight vector $\vec{r}$. The lowest weight condition reads

$$\begin{aligned} V_n^j|\vec{r}> &= 0, \qquad n>0 \\ V_0^j|\vec{r}> &= \sum_{a=1}^{m} f^j(r_a)|\vec{r}>, \end{aligned} \tag{6.4.1}$$

where the f^j are j'th order polynomials. The charge and Virasoro eigenvalues are given by

$$\begin{aligned} V_0^0|\vec{r}> &= \sum_{a=1}^{m} r_a\,|\vec{r}> \\ V_0^1|\vec{r}> &= \sum_{a=1}^{m} \frac{r_a^2}{2}\,|\vec{r}>\,. \end{aligned} \tag{6.4.2}$$

It was also found that these representations have strong relations with the unitary, irreducible representations of the algebra $\widehat{U(1)}^{\otimes m}$. If we denote by $D(g, c, \vec{r})$ the irreducible, unitary representation of the algebra g with weight vector $\vec{r}$ and central charge c, then these relations read

$$\begin{aligned} D(W_{1+\infty}, 1, r) &\sim D\left(\widehat{U(1)}, 1, r\right) \\ D(W_{1+\infty}, m, \vec{r}) &\sim D\left(\widehat{U(1)}^{\otimes m}, m, \vec{r}\right) \quad \textit{if}\ \forall_{a\neq b}\ (r_a - r_b) \notin \mathbb{Z} \\ D(W_{1+\infty}, m, \vec{r}) &\subset D\left(\widehat{U(1)}^{\otimes m}, m, \vec{r}\right) \quad \textit{if}\ \exists_{a\neq b}\ (r_a - r_b) \in \mathbb{Z}. \end{aligned} \tag{6.4.3}$$

The $\sim$ means one-to-one equivalence. There are two distinct classes of representations, *generic* and *degenerate*

- The $W_{1+\infty}$ representations with non-integer weight differences $(r_a - r_b)\ \forall_{a\neq b}$ are called *generic*. They are one-to-one equivalent to $\widehat{U(1)}^{\otimes m}$ representations. The fusion rule for combining these representations is just the addition of weight vectors. The basis vectors $\{\vec{v}^1, \cdots, \vec{v}^m\}$ spanning the weight lattice in a generic $W_{1+\infty}$ theory have to satisfy
$$\forall_I \forall_{a\neq b}\ (\vec{v}^I)_a - (\vec{v}^I)_b \notin Q \tag{6.4.4}$$
- A $W_{1+\infty}$ representation that contains at least one integer weight difference is called *degenerate*. In this case, an irreducible $\widehat{U(1)}^{\otimes m}$ representation is *reducible* with

respect to $W_{1+\infty}$. The fusion rule for combining degenerate representations is more complicated than simple addition of weight vectors. A $W_{1+\infty}$ theory that contains only degenerate representations is called a *minimal model.*

A degenerate weight vector can be grouped into so-called 'congruence classes',

$$\begin{aligned} \{r_1, \cdots, r_m\} &= \{s_1 + n_1^{(1)}, \cdots, s_1 + n_{m_1}^{(1)}\} \cup \cdots \cup \{s_k + n_1^{(k)}, \cdots, s_k + n_{m_k}^{(k)}\}; \\ n_j^{(i)} \in \mathbb{Z} \quad &, \quad \sum_{i=1}^{k} m_i = m, \quad (s_i - s_j) \notin \mathbb{Z}\, \forall_{i \neq j}. \end{aligned} \tag{6.4.5}$$

A representation with k congruence classes is the tensor product of k one-class representations.

6.4.2 Physical interpretation

The above presented mathematical results have important physical implications. The fact that the central charge can only have integer values suggests that m-fluid systems are the only possible physical realisations of $W_{1+\infty}$ in the QHE. Furthermore, all eigenvalues of the Cartan subalgebra V_0^j (6.4.1) consist of parts that depend only on one component of the weight vector; this indicates that the fluids are noninteracting.

The relation (6.4.3) between the representations of $W_{1+\infty}$ and $\widehat{U(1)}^{\otimes m}$ enables us to make contact with the effective edge theory. The currents on the edges of the noninteracting incompressible fluids are identified with the decoupled fields γ_a defined in (5.4.10). The physical parameters of the m-fluid system are *not* given by the eigenvalues of the decoupled operators, but by linear combinations thereof, specified by a K-matrix in the same way as in (5.4.11, 5.4.13). However, the construction now goes in the opposite direction; we start with weight vectors in the decoupled basis and end up with a K-matrix. If the Hilbert space is built from a collection of lowest weight states $|\vec{r}>$ of the form

$$\vec{r} = \sum_{I=1}^{m} n_I \vec{v}^I, \tag{6.4.6}$$

where the n_I are integers, then the K-matrix follows from

$$(K^{-1})^{IJ} = \vec{v}^I \cdot \vec{v}^J := \sum_{a=1}^{m} \kappa_a (\vec{v}^I)_a (\vec{v}^J)_a. \tag{6.4.7}$$

We now assign values ± 1 to the κ_a by hand. The physical properties (charge, statistics, filling fraction) of the states $|\vec{r}>$ follow from the K-matrix as in (5.4.24) and (5.4.6).

It can now be seen that not all basis vectors $\vec{v}^I$ are allowed in a theory of incompressible *electron* fluids. In order to describe a theory of electrons, the K-matrix constructed from the set $\{\vec{v}^1, \cdots, \vec{v}^m\}$ has to be integer with odd diagonal elements (see (5.4.20)). Vectors $\vec{v}^I$ that do not obey this constraint *do* yield a $W_{1+\infty}$ representation, but not one that can be used for the physics of the quantum Hall effect.

6.4.3 Generic and minimal theories; Stabilities

The complete Hilbert space for an m-fluid system is constructed by letting the V_n^j $(n < 0)$ act on a lattice Γ of lowest weight states $|r_1, \cdots, r_m>$ that is closed under the fusion rules.

This construction is very simple when the weight vectors $\vec{r}$ lead to generic representations. The lattice Γ then has the simple form

$$\Gamma = \{\vec{r} \mid \vec{r} = \sum_{I=1}^{m} n_I \vec{v}^I\} \tag{6.4.8}$$

where the basis vectors $\vec{v}^I$ satisfy (6.4.4). For example, for the experimentally observed $\nu = \frac{4}{3}$ state we have

$$\begin{aligned} K = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \quad &; \quad \nu = 1 + \frac{1}{3} = \frac{4}{3} \\ \vec{v}^1 = (1,0) \quad &; \quad \vec{v}^2 = (0, \frac{1}{\sqrt{3}}) \\ \Gamma = \{\vec{r} \mid \vec{r} = (n_1, \frac{n_2}{\sqrt{3}})\} &\qquad n_1, n_2 \in \mathbb{Z}. \end{aligned} \tag{6.4.9}$$

As is easily verified, the weight differences for all lattice elements are non-integer.

However, the Hilbert space for most of the physically interesting states is harder to construct, since one has to use degenerate representations. (See [27] for the fusion rules and lattice constructions.) In the case of the first Jain level fractions $\nu = \frac{m}{2pm+1}$, for example, we have

$$\begin{aligned} K_{IJ} = 2p + \delta_{IJ} \quad &; \quad (K^{-1})^{IJ} = \delta^{IJ} - \frac{2p}{2pm+1} \\ \nu &= \sum_{I,J=1}^{m} (K^{-1})^{IJ} = \frac{m}{2pm+1} \\ (\vec{v}^I)_a &= \frac{1}{m}\left(\frac{1}{\sqrt{2pm+1}} - 1\right) + \delta^I_a. \end{aligned} \tag{6.4.10}$$

Weight vectors that are obtained from these $\vec{v}^I$ are degenerate and consist of *one* congruence class, which is the *maximum* degeneracy that a weight vector can have.

This led CZT [27] to propose that the stability of a filling fraction is related to the amount of degeneracy in the $W_{1+\infty}$ representations used to construct the Hilbert space for that filling fraction; They argued that the stability occurs because the number of independent states in Hilbert space is much smaller for degenerate representations than for generic representations.

More work needs to be done in order to verify this idea.

Chapter 7

Conclusions

> There was a point to this story,
> but it has temporarily escaped the chronicler's mind.
>
> Douglas Adams, *So Long and Thanks for All the Fish*

We have looked at three different ways in which the quantum Hall effect can be described:

- a bottom-up approach: a microscopic description in terms of a hamiltonian and electron wavefunctions.
- a phenomenological model: the effective topological Chern-Simons gauge theory for the macroscopic currents, reducing to a 1+1 dimensional conformal field theory on the edge.
- an algebraic approach: $W_{1+\infty}$ representation theory, inspired on the phenomenological fact of incompressibility.

We have found a large similarity in these approaches. What they all have in common is the aspect of incompressibility, albeit in different guises.

In the microscopic approach, the algebra of area-preserving diffeomorphisms is a symmetry algebra of the hamiltonian. The enveloping algebra of the b and $b^\dagger$ operators forms a representation of $W_{1+\infty}$. Attaching an even number of flux quanta to electrons does not change this property. The many-electron ground states are incompressible and obey the lowest weight conditions of $W_{1+\infty}$, thereby defining specific representations of this algebra.

In the phenomenological approach the very equation of motion for the electric current yields incompressibility at the classical level. After quantisation of the Chern-Simons theory it is seen that the incompressibility takes the form of a constraint. The pure Chern-Simons action can be rewritten as a bosonic action on the edge, which is consistent with the fact that in the thermodynamic limit $W_{1+\infty}$ acts only on the edge, creating edge waves. The symmetry algebras of the edge boson theory are the Kač-Moody and Virasoro algebra, which are subalgebras of $W_{1+\infty}$. A very interesting result of the algebraic approach is that lowest weight representations $D(\widehat{U(1)}^{\otimes m}, m, \vec{r})$ of the Kač-Moody algebra can be expressed in one or more lowest weight representations $D(W_{1+\infty}, m, \vec{s})$ of $W_{1+\infty}$, showing that the commuting copies of $W_{1+\infty}$ correspond to the decoupled currents in the edge theory. The interactions between the currents in the Chern-Simons action and between the quantum fluids in the $W_{1+\infty}$ representations are given by exactly the same K-matrix.

From the classification point of view, the results of all three approaches are roughly the same and can be summarised as follows

- The quantum Hall effect occurs when a number m of incompressible quantum fluids is formed, each of which can contain its own kind of composite fermion.
- All global physical parameters of a quantum Hall state are contained in a symmetric, invertible $m \times m$ matrix K. This has to be an integer matrix with odd diagonal elements, otherwise the theory cannot describe fermionic particles with elementary charge. The physical parameters characterising a Hall state are the filling fraction ν and the spectrum of (quasi)particle charges Q and statistics θ, obtained from the K-matrix as follows,

$$\nu = \sum_{I,J=1}^{m} (K^{-1})^{IJ} \quad ; \quad Q^I = (K^{-1})^{IJ} n_J$$
$$\frac{\theta_{\vec{Q}}}{\pi} = (K^{-1})^{IJ} n_I n_J \quad ; \quad n_L \in Z.$$

  The particle spectrum consists of composite fermions and fractionally charged anyons.
- The Hall fractions with the widest plateaus are given by the first level of Jain's hierarchy, characterised by

$$K_{IJ} = 2p \pm \delta_{IJ} \quad ; \quad \nu = \frac{m}{2pm \pm 1}.$$

There are of course differences between the various approaches. The effective current theory does not offer any clues as to the width of plateaus beyond the first Jain level, while $W_{1+\infty}$ representation theory does. The width of the plateaus may be proportional to the degeneracy of the $W_{1+\infty}$ representations.

An important point to bear in mind is that the Chern-Simons theory and the algebraic analysis of symmetries present in the QHE are of a phenomenological nature. They can tell us what the properties of quantum Hall states are, *not* how those properties arise from a quantum theory of two-dimensional electrons in a random potential and a strong magnetic field.

Our understanding of the (fractional) quantum Hall effect is far from complete. One of missing links is a quantum theory explaining the dynamics of incompressibility and flux binding from the microscopic physics. (Important work in this direction was recently done by Pruisken et al. [29].)

Another long standing question is how to predict the stabilities of fractions outside the first level of Jain's hierarchy. The answer may come from a classification of $W_{1+\infty}$ representation degeneracies and symmetries of the K-matrix and weight lattices. (A recent attempt was made by Fröhlich et al. [30].) On the other hand, a full microscopic theory may be necessary for a good understanding.

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